Spaces of primitive elements in dual modules over Steenrod algebra

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Presented work studies subspace of all primitives $PB(n)$ in $B(n) = (A(n-1)/A(n))^*$, with respect to the coaction $\phi_n^* : B(n) \to A^* \otimes B(n)$, where the primitives are those elements $\alpha \in B(n)$ for which $\phi_n^*(\alpha) = 1 \otimes \alpha$, here A^* is dual Steenrod algebra. Modules $B(n)$ were studied in [5, 6, 7]. Works [1, 2, 3, 4, 7] contain all useful notions and results: [3] studies structures of modules over Steenrod algebra A and their duals in dual Steenrod algebra A^* [2, 4]; [1] studies modules $A(n)$ over A generated by annihilators of cohomology classes with degrees no greater then n and $A(n)^*$ is the corresponding dual module of $A(n)$; In [5, 7], $A(n)^+$ is defined as annihilators of cohomology operations with excess greater then n, proved that as vector space over Z_p it is generated by all monomials of A^* with multiplicity no greater then n and $A(n)^+$ can be considered both left and right comodule over A^*. The filtration described in [6] Theorem 1 property 2 and 3 yields $PB(n) = \bigcup_t PB(n)_t$ and $PB(n)_t = \bigoplus_s PB(n)_t^s$, where s is the number of τ operations and where t is the biggest index of such operations. By property 4 Theorem 1 follows: $P(B(n)_t/B(n)_{t-1}) \simeq PB(n-1)_{t-1}$. Found properties of $PB(n)_t$ can be shortly summarized here:

Theorem 1 (**Properties of $PB(n)_t$ spaces**). *(1) $PB(n) = \bigcup_t PB(n)_t$ and the following diagram is commutative with exact rows and columns:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & PB(n)_{t-1} & \xhookrightarrow{i_{t-1}} & PB(n)_t & \xrightarrow{\pi_t} & PB(n-1)_{t-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B(n)_{t-1} & \xhookrightarrow{i_{t-1}} & B(n)_t & \xrightarrow{\pi_t} & B(n-1)_{t-1} & \longrightarrow & 0 \\
 & & \downarrow \phi_n & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \\
0 & \longrightarrow & A^* \bigotimes B(n)_{t-1} & \xhookrightarrow{Id\otimes i_{t-1}} & A^* \bigotimes B(n)_t & \xrightarrow{Id\otimes\pi_t} & A^* \bigotimes B(n-1)_{t-1} & \longrightarrow & 0
\end{array}$$

Here all denoted maps are induced by maps defined in Theorem 1 property 4 [6].

(2) For even n: $PB(n)_{-1} = PB(n)_0 = \langle \xi_1^{n/2} \rangle$; for odd n: $PB(n)_0 = \langle \xi_1^{\frac{n-1}{2}} \tau_0 \rangle$.

(3) $PB(n)_t = \bigoplus_s PB(n)_t^s$ and the following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccc}
PB(n)_{t-1}^s & \xhookrightarrow{i_{t-1}^s} & PB(n)_t^s & \xrightarrow{\pi_t^s} & PB(n-1)_{t-1}^{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
B(n)_{t-1}^s & \xhookrightarrow{i_{t-1}^s} & B(n)_t^s & \xrightarrow{\pi_t^s} & B(n-1)_{t-1}^{s-1} \\
\downarrow \phi_n & & \downarrow \phi_n & & \downarrow \phi_{n-1} \\
A^* \bigotimes B(n)_{t-1}^s & \xhookrightarrow{Id\otimes i_{t-1}^s} & A^* \bigotimes B(n)_t^s & \xrightarrow{Id\otimes\pi_t^s} & A^* \bigotimes B(n-1)_{t-1}^{s-1}
\end{array}$$

Here all denoted maps are induced ones as in (1).

(4) Composition of maps

$$PB(n)_t^{s,deg} \hookrightarrow B(n)^{s,deg} \twoheadrightarrow B(n)^{s,deg}/(I \cap B(n)^{s,deg})$$

is monomorphism, here $I = \langle \xi_2, \xi_3, \xi_4, \cdots \rangle$ *is ideal in* A^**, where deg is degree. The dimension of* $B(n)^{s,deg}/(I \cap B(n)^{s,deg})$ *is less or equal 1.*

(5) $dim(PB(n)^{s,deg}_t) \leq 1$.

(6) For map $PB(2m+1)_t \xrightarrow{\pi_t} PB(2m)_{t-1}$, $\xi_1^m \in Im(\pi_0)$ *and for* $t > 0$: $\xi_1^m \notin Im(\pi_t)$*; if* $s > 1$ *and* $\alpha \in PB(n)^s_t$ *then* $\alpha = \alpha' \tau_0$.

(7) Denote $J(n)^{s,deg} = B(n)^{s,deg}/(I \cap B(n)^{s,deg})$ *then the following diagram is commutative:*

$$\begin{array}{ccc}
PB(n)^{s,deg}_t & \xrightarrow{\pi_t^{s,deg}} & PB(n-1)^{s-1,deg-|\tau_t|}_{t-1} \\
\downarrow & & \downarrow \\
B(n)^{s,deg}_t & \xrightarrow{\pi_t^{s,deg}} & B(n-1)^{s-1,deg-|\tau_t|}_{t-1} \\
\downarrow j_n & & \downarrow j_{n-1} \\
J(n)^{s,deg} & & J(n-1)^{s-1,deg-|\tau_t|}
\end{array}$$

These properties may be useful to find bases of $PB(n)^{s,deg}_t$ spaces and $PB(n)$ basis as result.

References

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