The behaviour of weak solutions of boundary value problems for linear elliptic second order equations in unbounded cone - like domains

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Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$ with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded domain with the smooth boundary ∂G. We assume that there exists $R \gg 1$ such that $G = G_0 \cup G_R$, where G_0 is a bounded domain in $\mathbb{R}^n$, $G_R = \{x = (r, \omega) \in \mathbb{R}^n |\ r \in (R, \infty)$, $\omega \in \Omega \subset S^{n-1}$, S^{n-1} is the unit sphere$\}$.

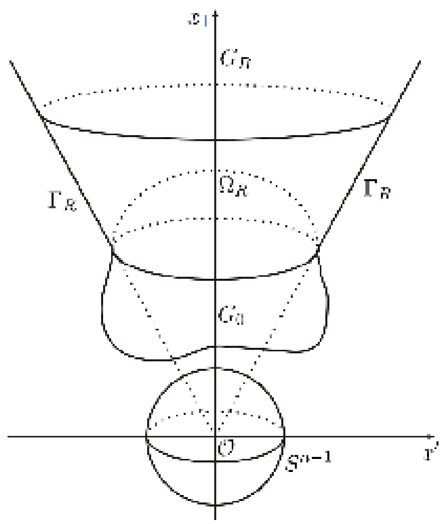

FIGURE 0.1. An unbounded cone-like domain

We consider the following linear problem :

$$\begin{cases} \frac{\partial}{\partial x_i}\left(a^{ij}(x)u_{x_j}\right) + b^i(x)u_{x_i} + c(x)u = f(x), & x \in G; \\ \alpha(x)\frac{\partial u}{\partial \nu} + \frac{1}{|x|}\gamma\left(\frac{x}{|x|}\right)u = g(x), & x \in \partial G; \\ \lim\limits_{|x|\to\infty} u(x) = 0; \end{cases} \tag{L}$$

here $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\vec{n}, x_i)\frac{\partial}{\partial x_j}$, where $\vec{n}$ denotes the unit outward with respect to G normal to ∂G, $\gamma(\omega)$ is positive bounded piecewise smooth function on $\partial\Omega$ such that $\gamma(\omega) \geq \gamma_0 > 0$ and $\alpha(x) = \begin{cases} 0, & \text{if } x \in \mathcal{D}; \\ 1, & \text{if } x \notin \mathcal{D}, \end{cases}$ $\mathcal{D} \subseteq \partial G$ is the part of the boundary ∂G, where the Dirichlet boundary condition is posed.

We assume that the following conditions are fulfilled :

(a) the condition of uniform ellipticity :

$$\nu\xi^2 \leq a^{ij}(x)\xi_i\xi_j \leq \mu\xi^2 \quad \forall x \in \overline{G}, \quad \forall \xi \in \mathbb{R}^n;$$
$$\nu, \mu = const > 0 \ \textit{and} \ \lim_{|x|\to\infty} a^{ij}(x) = \delta_i^j$$

;

(b) $a^{ij}(x) \in C^0(\overline{G}),\ b^i(x) \in L_p(G),\ p > n$; *for them holds inequalities*

$$\sqrt{\sum_{i,j=1}^{n} |a^{ij}(x) - \delta_i^j|^2} \leq \mathcal{A}\left(\frac{1}{|x|}\right), \quad |x| \left(\sum_{i=1}^{n} |b^i(x)|^2\right)^{\frac{1}{2}} \leq \mathcal{A}\left(\frac{1}{|x|}\right), \quad x \in G_R,$$

where $\mathcal{A}(t),\ t \geq 0$ *is a monotonically increasing, nonnegative function, continuous at zero and* $\lim\limits_{r\to\infty} \mathcal{A}\left(\frac{1}{r}\right) = 0$;

(c) $0 \geq c(x) \in L_{p/2}(G) \cap L_2(G)$;

(d) $f(x) \in L_{p/2}(G) \cap L_2(G)$;

(e) *there exist numbers* $f_1 \geq 0,\ g_1 \geq 0,\ s > 0$ *such that* $|f(x)| \leq f_1|x|^{-s-2},\ |g(x)| \leq g_1|x|^{-s-1}$.

In [1] we investigated the behaviour of weak solutions to the problem (L) in a neighborhood of infinity assuming that the function $\mathcal{A}(t)$ is Dini-continuous at zero in the meaning that the integral $\int\limits_0^d \frac{\mathcal{A}(t)}{t}dt$ is finite for some $d > 0$. More precisely, we obtained the following theorem:

Theorem 1. *If the function* $\mathcal{A}(t)$ *from assumption* (b) *is Dini-continuous at zero, then*

$$|u(x)| \leq C_0 \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1\right) \cdot \begin{cases} |x|^{\lambda_-}, & \text{if } s > -\lambda_-; \\ |x|^{\lambda_-} \ln|x|, & \text{if } s = -\lambda_-; \\ |x|^{-s}, & \text{if } 0 < s < -\lambda_-, \end{cases}$$

where $\lambda_- = \frac{2-n-\sqrt{(n-2)^2+4\vartheta}}{2}$, ϑ *is the smallest positive eigenvalue of the eigenvalue problem for the Laplace-Beltrami operator on the unit sphere.*

In [3] our aim was to derive the estimate of the weak solution modulus for our problems near the infinity under assumption that leading coefficients of the equations **do not** satisfy the Dini-continuity condition. We obtained the following result:

Theorem 2. *Let* $u(x)$ *be a weak solution of problem* (L) *and assumptions (a)-(e) be satisfied with* $\mathcal{A}(t)$ *which is* **not Dini-continuous** *at zero. Then for all* $\varepsilon > 0$ *there are* $R \gg 1$ *and a constant* $C_\varepsilon > 0$ *such that for all* $x \in G_R$

$$|u(x)| \leq C_\varepsilon \left(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\gamma_0}} g_1\right) \cdot \begin{cases} |x|^{\lambda_- + \varepsilon}, & \text{if } s \geq -\lambda_-; \\ |x|^{-s}, & \text{if } 0 < s < -\lambda_-. \end{cases}$$

References

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