A universal coregular countable second-countable space

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A Hausdorff topological space X is called *superconnected* (resp. *coregular*) if for any nonempty open sets $U_1, \dots U_n \subseteq X$, the intersection of their closures $\overline{U}_1 \cap \dots \cap \overline{U}_n$ is not empty (resp. the complement $X \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_n)$ is a regular topological space). A canonical example of a coregular superconnected space is the projective space $\mathbb{Q}\mathsf{P}^\infty$ of the topological vector space

$$\mathbb{Q}^{<\omega} = \{(x_n)_{n\in\omega} \in \mathbb{Q}^\omega : |\{n \in \omega : x_n \neq 0\}| < \omega\}$$

over the field of rationals $\mathbb{Q}$. The space $\mathbb{Q}\mathsf{P}^\infty$ is the quotient space of $\mathbb{Q}^{<\omega} \setminus \{0\}^\omega$ by the equivalence relation $x \sim y$ iff $\mathbb{Q}{\cdot}x = \mathbb{Q}{\cdot}y$.

We prove that every countable second-countable coregular space is homeomorphic to a subspace of $\mathbb{Q}\mathsf{P}^\infty$, and a topological space X is homeomorphic to $\mathbb{Q}\mathsf{P}^\infty$ if and only if X is countable, second-countable, and admits a decreasing sequence of closed sets $(X_n)_{n\in\omega}$ such that (i) $X_0 = X$, $\bigcap_{n\in\omega} X_n = \emptyset$, (ii) for every $n \in \omega$ and a nonempty open set $U \subseteq X_n$ the closure $\overline{U}$ contains some set X_m, and (iii) for every $n \in \omega$ the complement $X \setminus X_n$ is a regular topological space.

Using this topological characterization of $\mathbb{Q}\mathsf{P}^\infty$ we find topological copies of the space $\mathbb{Q}\mathsf{P}^\infty$ among quotient spaces, orbit spaces of group actions, and projective spaces of topological vector spaces over countable topological fields.

References

[1] T. Banakh, Ya. Stelmakh, *A universal coregular countable second-countable space*, preprint (arxiv.org/abs/2003.06293).