Trace Regularization Problem On a Banach Space

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Let $\mathcal{H}$ be a separable Hilbert space and let $S_1[\mathcal{H}]$ be the trace class operators on $\mathcal{H}$ (First Schatten Class, [3]).

Consider $\mathcal{H}_1 = L^2(\mathcal{H};\ [0,\pi])$ and define an inner product on $\mathcal{H}_1$ by:

$$(f,g)_{\mathcal{H}_1} = \int_0^\pi (f(t),g(t))_{\mathcal{H}} dt$$

for all $f,g \in \mathcal{H}_1$.

- With this inner product, $\mathcal{H}_1$ is also a separable Hilbert space.

Here, we study the same problem in [2], with $\mathcal{H}$ replaced by a arbitrary separable Banach space $\mathcal{B}$, under the following conditions:

(1) $Q(t)$ has a weak second-order derivative in $[0,\pi]$ and for $t \in [0,\pi]$, $Q^{(i)}(t)$ $(i = 0,1,2)$ is a self adjoint trace class operator on $\mathcal{B}$.
(2) $\|Q\|_{\mathcal{H}_1} < 1$.
(3) $\mathcal{H}_1$ has an o.n.b. $\{\varphi_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty \|Q\varphi_n\|_{\mathcal{H}_1} < \infty$.
(4) $\left\|Q^i(t)\right\|_{S_1[\mathcal{B}]}$ $(i = 0,1,2)$ is a bounded measurable function on $[0,\pi]$.

It is clear from (4), that this is a nontrivial problem since, among other things, in the standard approach, there are a number of possible definitions of $S_1[\mathcal{B}]$ (see [3] and Pietsch [9]).
We assume that $\mathcal{B}$ is a continuous dense embedding in a separable Hilbert space $\mathcal{H}$ and for each $f,g \in \mathcal{B}$, $(f,g)_h = (f,g)_{\mathcal{H}}$ is the Hilbert functional on $\mathcal{B}$.

Theorem 1 (Polar Representation Theorem). *Let $\mathcal{B}$ be a separable Banach space. If $A \in \mathbb{C}[\mathcal{B}]$, then there exists a partial isometry U and a self-adjoint operator T, with $D(T) = D(A)$ and $A = UT$. Furthermore, $T = [A^*A]^{1/2}$ in a well-defined sense.*

Def. If $A \in \mathbb{S}_1[\mathcal{B}]$, we called it a trace class (or nuclear) operator on $\mathcal{B}$.
  * Since $\mathbb{S}_p[\mathcal{H}]$ is a two sided *ideal, it follows that the same is true for $\mathbb{S}_p[\mathcal{B}]$.
  * For $1 \le p < \infty$, $A \in \mathbb{S}_p[\mathcal{B}]$ and $B \in \mathcal{L}[\mathcal{B}]$ then $AB, BA \in \mathbb{S}_p[\mathcal{B}]$ and

$$\|AB\|_{\mathbb{S}_p[\mathcal{B}]} \le \|B\|_{\mathcal{L}[\mathcal{B}]} \|A\|_{\mathbb{S}_p[\mathcal{B}]}$$
$$\|BA\|_{\mathbb{S}_p[\mathcal{B}]} \le \|B\|_{\mathcal{L}[\mathcal{B}]} \|A\|_{\mathbb{S}_p[\mathcal{B}]}$$

Lemma 2. *If $\lambda \notin \sigma(L_0)$ then $QR_0(\lambda) \in \mathbb{S}_1[\mathcal{H}_1]$*

Lemma 3. *The operator valued function $R(\lambda) - R_0(\lambda)$ is analytic in $\rho(L)$, the resolvent set of L, with respect to the $\mathbb{S}_1[\mathcal{H}_1]$ norm.*

Theorem 4. *The regularized trace formula for operator L on B with the conditions on operator function $Q(t)$ is given by*

$$\sum_{m=0}^\infty \left[\sum_{n=1}^\infty (\lambda_{mn} - \mu_m) - \frac{1}{\pi}\int_0^\pi trQ(t)dt\right] = \frac{1}{4}[tr(Q(0) + trQ(\pi)$$

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