Centrally extended generalization of the superconformal loop Lie algebra and integrable heavenly type systems on supermanifolds

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Let us consider the semi-direct sum $\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg}$ of the loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^{1|N})$, consisting of the superconformal vector fields on a supertor $\mathbb{T}^{1|N}$ in the forms:

$$\tilde{a} := a\partial/\partial x + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}a)D_{\vartheta_i}, \quad a := a(x,\vartheta;\lambda), \tag{1}$$

where $a \in C^\infty(\mathbb{T}^{1|N} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_0)$, $(x,\vartheta) \in \mathbb{T}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1^N$, $\Lambda := \Lambda_0 \oplus \Lambda_1$ is a infinite-dimensional Grassmann algebra over $\mathbb{C} \supset \Lambda_0$, $\vartheta := (\vartheta_1, \vartheta_2, \ldots, \vartheta_N)$ and $D_{\vartheta_i} := \partial/\partial\vartheta_i + \vartheta_i\partial/\partial x$, $i = \overline{1,N}$, which are holomorphic in the "spectral" parameter $\lambda \in \mathbb{C}$ on the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and exterior $\mathbb{D}^1_- \subset \mathbb{C}$ regions of the unit centrally located disk $\mathbb{D}^1 \subset \mathbb{C}$, and its regular dual space $\tilde{\mathcal{G}}^*_{reg}$ with respect to the parity:

$$(\tilde{l},\tilde{a})_0 = \operatorname{res}\lambda^{-1}\int_{\mathbb{T}^{1|N}} dx d^N\vartheta\,(la), \quad \tilde{l} := l(x,\vartheta;\lambda)(dx + \sum\nolimits_{i=1}^{N}\vartheta_i d\vartheta_i) \in \tilde{\mathcal{G}}^*_{reg}, \tag{2}$$

where $l \in C^\infty(\mathbb{T}^{1|(2k-1)} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_1)$ if $N = 2k-1$ and $l \in C^\infty(\mathbb{T}^{1|2k} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_0)$ if $N = 2k$, $k \in \mathbb{N}$. The superconformal loop Lie algebra $\tilde{\mathcal{G}}$ possesses the commutator:

$$[\tilde{a},\tilde{b}] = \tilde{c}, \quad \tilde{c} := c\partial/\partial x + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}c)D_{\vartheta_i},$$

$$c := a(\partial b/\partial x) - b(\partial a/\partial x) + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}a)(D_{\vartheta_i}b), \quad \tilde{a},\tilde{b} \in \tilde{\mathcal{G}},$$

splits into the direct sum of its Lie subalgebras $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, for which the following dual spaces are identified: $\tilde{\mathcal{G}}^*_{+,reg} \simeq \tilde{\mathcal{G}}_-$, $\tilde{\mathcal{G}}^*_{-,reg} \simeq \tilde{\mathcal{G}}_+$. Here $\tilde{a}(\infty) = 0$ for any $\tilde{a}(\lambda) \in \tilde{\mathcal{G}}_-$. On $\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg}$ one determines the commutator:

$$[\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes \tilde{m}] := [\tilde{a},\tilde{b}] \ltimes (ad^*_{\tilde{a}}\tilde{m} - ad^*_{\tilde{b}}\tilde{l}), \quad \tilde{a},\tilde{b} \in \tilde{\mathcal{G}}, \quad \tilde{l},\tilde{m} \in \tilde{\mathcal{G}}^*_{reg},$$

where ad^* is the co-adjoint action of $\tilde{\mathcal{G}}$ with respect to the parity (2) and

$$ad^*_a l = l_x a + \frac{4-N}{2} l a_x + \frac{(-1)^{N+1}}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}l)(D_{\vartheta_i}a)$$

for any vector field $\tilde{a} \in \tilde{\mathcal{G}}$ and a fixed element $\tilde{l} \in \tilde{\mathcal{G}}^*_{reg}$, as well as nondegenerate symmetric bilinear form:

$$(\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes m) = (\tilde{l},\tilde{b})_0 + (\tilde{m},\tilde{a})_0.$$

One constructs the central extension $\hat{\mathfrak{G}} := \tilde{\mathfrak{G}} \oplus \mathbb{C}$ of the Lie algebra $\tilde{\mathfrak{G}} := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg})$ by the 2-cocycle [1]:

$$\omega_2(\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes m) = \int_{\mathbb{S}^1} dz((\tilde{l},\partial\tilde{b}/\partial z)_0 - (\tilde{m},\partial\tilde{a}/\partial z)_0), \quad (\tilde{a} \ltimes \tilde{l}),(\tilde{b} \ltimes \tilde{m}) \in \tilde{\mathfrak{G}}, \quad z \in \mathbb{S}^1.$$

The Lie algebra $\tilde{\mathfrak{G}}$ permits the standard splitting $\tilde{\mathfrak{G}} := \tilde{\mathfrak{G}}_+ \oplus \tilde{\mathfrak{G}}_-$ of the Lie algebra $\tilde{\mathfrak{G}}$ into the direct sum of its Lie subalgebras $\tilde{\mathfrak{G}}_+ := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}}_+ \ltimes \tilde{\mathcal{G}}^*_{-,reg})$ and $\tilde{\mathfrak{G}}_- := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}}_- \ltimes \tilde{\mathcal{G}}^*_{+,reg}))$. Thus, by means of the $\mathcal{R}$-operator approach [2] one introduces the following Lie-Poisson bracket:

$$\begin{aligned}\{\mu,\nu\}_{\mathcal{R}} = {} & (\tilde{a} \ltimes \tilde{l}, [R\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), \nabla_l\nu(\tilde{a} \ltimes \tilde{l})] + [\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), R\nabla_l\nu(\tilde{a} \ltimes \tilde{l})]) + \\ & + \omega_2(R\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), \nabla_l\nu(\tilde{a} \ltimes \tilde{l})) + \omega_2(\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), R\nabla_l\nu(\tilde{a} \ltimes \tilde{l})),\end{aligned} \tag{3}$$

where $\mu, \nu \in \mathcal{D}(\tilde{\mathfrak{G}}^*)$ are arbitrary smooth by Frechet functionals on $\tilde{\mathfrak{G}}^*$, $\mathcal{R} = (P_+ - P_-)/2$, P_+ and P_- are projectors on $\tilde{\mathfrak{G}}_+$ and $\tilde{\mathfrak{G}}_-$ respectively, on the dual space $\tilde{\mathfrak{G}}^* \simeq \tilde{\mathfrak{G}}$ to the Lie algebra $\tilde{\mathfrak{G}}$. Here $\nabla_l h(\tilde{a} \ltimes \tilde{l}) := (\nabla_l h_{\tilde{l}} \ltimes \nabla_l h_{\tilde{a}}) \in \tilde{\mathfrak{G}}$ and $\nabla_r h(\tilde{a} \ltimes \tilde{l}) := (\nabla_r h_{\tilde{l}} \ltimes \nabla_r h_{\tilde{a}}) \in \tilde{\mathfrak{G}}$ are left and right gradients of any smooth functional $h \in \mathcal{D}(\tilde{\mathfrak{G}}^*)$ at a point $(\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*$. Due to the Adler-Kostant-Symes theory [2] the Lie-Poisson bracket (3) generates the hierarchy of Hamiltonian flows:

$$\partial(\tilde{a} \ltimes \tilde{l})/\partial t_p := -ad^*_{P_+ \nabla_l h^{(p)}(\tilde{a} \ltimes \tilde{l})}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(p)}\}_{\mathcal{R}}, \quad (\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*, \quad p \in \mathbb{Z}_+,$$

where $P_+ \nabla_l h^{(p)}(\tilde{a} \ltimes \tilde{l}) = (\nabla_l h^{(p)}_{\tilde{l},+} \ltimes \nabla_l h^{(p)}_{\tilde{a},+})$, $\nabla_l h^{(p)}(\tilde{a} \ltimes \tilde{l}) = \lambda^p \nabla_l h(\tilde{a} \ltimes \tilde{l})$, $\nabla_l h_{\tilde{l}} \sim \sum_{j \in \mathbb{Z}_+} \nabla_l h_{\tilde{l},j} \lambda^{-j}$ and $\nabla_l h_{\tilde{a}} \sim \sum_{j \in \mathbb{Z}_+} \nabla_l h_{\tilde{a},j} \lambda^{-j}$ as $|\lambda| \to \infty$, for any Casimir invariant $h \in I(\hat{\mathfrak{G}}^*)$, satisfying, by definition, the following relationship:

$$ad^*_{\nabla_l h(\tilde{a} \ltimes \tilde{l})}(\tilde{a} \ltimes \tilde{l}) = 0, \quad (\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*.$$

Any two Hamiltonian flows on $\tilde{\mathfrak{G}}^*$ in the forms:

$$\partial(\tilde{a} \ltimes \tilde{l})/\partial y = \{\tilde{a} \ltimes \tilde{l}, h^{(y)}(\tilde{a} \ltimes \tilde{l})\}_{\mathcal{R}}, \quad \partial(\tilde{a} \ltimes \tilde{l})/\partial t = \{\tilde{a} \ltimes \tilde{l}, h^{(t)}(\tilde{a} \ltimes \tilde{l})\}_{\mathcal{R}},$$

where $\nabla_l h^{(y)} = \lambda^{p_y} \nabla_l h(\tilde{a} \ltimes \tilde{l})$, $\nabla_l h^{(t)} = \lambda^{p_t} \nabla_l h(\tilde{a} \ltimes \tilde{l})$, $p_y, p_t \in \mathbb{Z}_+$, and $h \in I(\hat{\mathfrak{G}}^*)$, give rise to the separately commuting evolution equations:

$$\partial\tilde{a}/\partial y = -[\nabla_l h^{(y)}_{\tilde{l},+}, \tilde{a}] + \partial(\nabla_l h^{(y)}_{\tilde{l},+})/\partial z, \quad \partial\tilde{a}/\partial t = -[\nabla_l h^{(t)}_{\tilde{l},+}, \tilde{a}] + \partial(\nabla_l h^{(t)}_{\tilde{l},+})/\partial z, \tag{4}$$

and

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla_l h^{(y)}_{\tilde{l},+}} \tilde{l} + ad^*_{\tilde{a}} \nabla_l h^{(y)}_{\tilde{a},+} + \partial(\nabla_l h^{(y)}_{\tilde{a},+})/\partial z,$$

$$\partial\tilde{l}/\partial t = -ad^*_{\nabla_l h^{(t)}_{\tilde{l},+}} \tilde{l} + ad^*_{\tilde{a}} \nabla_l h^{(t)}_{\tilde{a},+} + \partial(\nabla_l h^{(t)}_{\tilde{a},+})/\partial z.$$

Proposition 1. *The commutativity of evolutions* (4) *is equivalent to the relationship:*

$$[\nabla_l h^{(y)}_{\tilde{l},+}, \nabla_l h^{(t)}_{\tilde{l},+}] - \partial(\nabla_l h^{(y)}_{\tilde{l},+})/\partial t + \partial(\nabla_l h^{(t)}_{\tilde{l},+})/\partial y = 0, \tag{5}$$

which is reduced on every coadjoint orbit of the Lie algebra $\hat{\mathfrak{G}}$ to the Lax-Sato representation for some system of nonlinear heavenly type equations on a functional supermanifold. The relationship (5) *is a compatibility condition for the following linear vector equations:*

$$\partial\psi/\partial y + \nabla_l h^{(y)}_{\tilde{l},+} \psi = 0, \quad \partial\psi/\partial z + \tilde{a}\psi = 0, \quad \partial\psi/\partial t + \nabla_l h^{(t)}_{\tilde{l},+} \psi = 0,$$

where $(y, t; \lambda, z, x, \theta) \in (\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^{1|N}))$ and $\psi \in C^2(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^{1|N})); \mathbb{C})$.

By use of the Lax-Sato compatibility condition (5) one can construct integrable systems of heavenly type equations on functional supermanifolds, which can be considered as generalizations of Lax-Sato integrable superanalogs [3] of the Mikhalev-Pavlov heavenly type equation, choosing the smooth functions $a := \sum_{k=1}^{K-1} w_{k,x}(x,\theta)\lambda^k - \lambda^K$ and $l := \sum_{k=1}^{K-1} \xi_{k,x}(x,\theta)\lambda^k$, $K \in \mathbb{N}$, in (1) and (2) respectively.

References

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