Dynamics and exact solutions of linear PDEs

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The report presents a new method for constructing exact solutions of the classical linear equations of mathematical physics of parabolic, hyperbolic, elliptic and variable types. The method is a generalization of the theory of finite-dimensional dynamics proposed for evolutionary differential equations [1, 5]. The theory of finite-dimensional dynamics is a natural development of the theory of dynamical systems. Dynamics make it possible to find families that depends on a finite number of parameters among all solutions of PDEs (see [2, 3]).

Consider the following class of second order linear partial differential equations

$$u_{tt} + 2b(x)u_{tx} + c(x)u_{xx} + h(x)u_t + g(x)u_x + f(x) = 0, \tag{1}$$

where b, c, h, g, f are functions of the class C^∞. Such equations are equivalent to the following evolutionary systems

$$\begin{cases} u_t = v, \\ v_t = -2b(x)v_x - c(x)u_{xx} - h(x)v - g(x)u_x - f(y). \end{cases} \tag{2}$$

We call the system of ordinary differential equations of order $k+1$

$$\begin{cases} y^{(k+1)} = Y\left(x, y, z, y', z' \ldots, y^{(k)}, z^{(k)}\right), \\ z^{(k+1)} = Z\left(x, y, z, y', z' \ldots, y^{(k)}, z^{(k)}\right) \end{cases} \tag{3}$$

a *dynamics* of equation (1) if the vector function

$$(\varphi, \psi) := (z_0, -2b(x)z_1 - c(x)y_2 - h(x)z_0 - g(x)y_1 - f(x))$$

is a generating function of infinitesimal characteristic symmetries of this system [4]. Here x, y_0, z_0, y_1, z_1, y_2, z_2 are canonical coordinates on the space of 2-jets $J^2(\mathbb{R}^1, \mathbb{R}^2)$.

Theorem 1. *The vector field on* $J^k(\mathbb{R}^1, \mathbb{R}^2)$

$$S = \varphi\frac{\partial}{\partial y_0} + \psi\frac{\partial}{\partial z_0} + \mathcal{D}(\varphi)\frac{\partial}{\partial y_1} + \mathcal{D}(\psi)\frac{\partial}{\partial z_1} + \cdots + \mathcal{D}^k(\varphi)\frac{\partial}{\partial y_k} + \mathcal{D}^k(\psi)\frac{\partial}{\partial z_k} \tag{4}$$

is an infinitesimal characteristic symmetry of system (3) if the following conditions hold:

$$\begin{cases} \mathcal{D}^{k+1}(\varphi) - S(Y) = 0, \\ \mathcal{D}^{k+1}(\psi) - S(Z) = 0. \end{cases} \tag{5}$$

Here

$$\mathcal{D} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y_0} + z_1\frac{\partial}{\partial z_0} + \cdots + y_k\frac{\partial}{\partial y_{k-1}} + z_k\frac{\partial}{\partial z_{k-1}} + Y\frac{\partial}{\partial y_k} + Z\frac{\partial}{\partial z_k}.$$

Let $\Gamma^k \subset J^2(\mathbb{R}^1, \mathbb{R}^2)$ be a k-graph of some solution of system (3) and let Φ_t be the shift along the vector field S. Then the surface $\Phi_t(\Gamma^k)$ is a k-graph of a solution of system (2).

Example 2. Consider the telegraph equation

$$u_{tt} - u_{xx} = au + bu_t + c, \tag{6}$$

where a, b, c are constants. This equation admits two types of dynamics:

$$\begin{cases} y_2 = \dfrac{y_1}{x+\alpha}, \\ z_2 = \dfrac{z_1}{x+\alpha} \end{cases} \tag{7}$$

and

$$\begin{cases} y_2 = \dfrac{2b\alpha - (x+\beta)\alpha^2}{4b^2 + 16a - \alpha^2(x+\beta)^2} \times y_1 - \dfrac{4\alpha}{4b^2 + 16a - \alpha^2(x+\beta)^2} \times z_1, \\ z_2 = -\dfrac{4a\alpha}{4b^2 + 16a - \alpha^2(x+\beta)^2} \times y_1 - \dfrac{2b\alpha + \alpha^2(x+\beta)}{4b^2 + 16a - \alpha^2(x+\beta)^2} \times z_1. \end{cases} \tag{8}$$

Here α, β are arbitrary constants. The general solution of equation (7) is

$$\begin{cases} y(x) = C_3 + C_4(x+\alpha)^2, \\ z(x) = C_1 + C_2(x+\alpha)^2, \end{cases} \tag{9}$$

and the general solution of equation (8) is

$$\begin{cases} y(x) = \dfrac{1}{2}C_2x^2 + C_3x + C_4, \\ z(x) = \dfrac{1}{8\alpha}\left(x(C_2\beta - C_3)(2\beta + x)\alpha^2 + + (8C_1 + 2bx^2C_2 + 4bC_3x)\alpha - 32\left(a + \dfrac{b^2}{4}\right)C_2x\right). \end{cases} \tag{10}$$

Here $C_1, \ldots, C_4$ are arbitrary constants. Applying the shift transformations Φ_t to the obtained general solutions, we obtain particular solutions of equation (6). For example, the function

$$\begin{aligned} u(t,x) = & -1 + \frac{1}{10}\left(\frac{5}{2}x^2 + 5 + (10x + 1 - t)\sqrt{5}\right)e^{-\frac{1}{2}(t\sqrt{5}-1)} + \\ & + \frac{1}{10}\left(\frac{5}{2}x^2 + 5 + (-10x - 1 + t)\sqrt{5}\right)e^{\frac{1}{2}(t\sqrt{5}-1)} \end{aligned} \tag{11}$$

is a solution of equation (6). It corresponds to solution (10) with $a = b = c = 1$, $\alpha = 1, \beta = 0$ and $C_1 = 0, C_2 = 1, C_3 = 0, C_4 = 0, C_5 = 0$.

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