On the squares of diffeomorphisms of surfaces

Iryna Kuznietsova
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: `kuznietsova@imath.kiev.ua`

Sergiy Maksymenko
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: `maks@imath.kiev.ua`

Let M be a surface and $\mathcal{D}(M)$ be the group of C^∞-diffeomorphisms of M. There is a natural right action of the group $\mathcal{D}(M)$ on the space of smooth functions $C^\infty(M,\mathbb{R})$ defined by the following rule: $(h,f)\longmapsto f\circ h$, where $h\in\mathcal{D}(M)$, $f\in C^\infty(M,\mathbb{R})$.

Thus, the *stabilizer* of f with respect to the action

$$\mathcal{S}(f)=\{h\in\mathcal{D}(M)\mid f\circ\ h=f\}$$

consists of f-preserving diffeomorphisms of M.

Endow $\mathcal{D}(M)$ with Whitney C^∞-topology and its subspaces $\mathcal{S}(f)$ with induced one. Denote by $\mathcal{S}_{\mathrm{id}}(f)$ the identity path component of $\mathcal{S}(f)$.

Definition 1. Denote by $\mathcal{F}(M)$ the space of smooth functions $f\in C^\infty(M,\mathbb{R})$ satisfying the following conditions:

(1) The function f takes constant value at each connected component of ∂M and has no critical points in ∂M.
(2) For every critical point z of f there is a local presentation $f_z\colon\mathbb{R}^2\to\mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2\to\mathbb{R}$ without multiple factors.

Definition 2. A smooth vector field F will be called Hamiltonian-like for $f\in\mathcal{F}(M)$ if the following conditions hold:

(1) $F(x)=0$ if and only if x is a critical point of f,
(2) f takes constant values on orbits of F,
(3) Let z be a critical point of f. Then there exists a local representation of f at z as a homogeneous polynomial $g\colon(\mathbb{R}^2,0)\to(\mathbb{R},0)$ without multiple factors such that in the same coordinates (x,y) near the origin 0 in $\mathbb{R}^2$ we have $F=-g'_y\frac{\partial}{\partial x}+g'_x\frac{\partial}{\partial y}$.

The smooth flow $\mathbf{F}\colon M\times\mathbb{R}\to M$ generated by a Hamiltonian-like vector field for f will be called Hamiltonian-like flow for f.

Denote by $\Delta^-(f)$ the set of diffeomorphisms from $\mathcal{S}(f)$ leaving invariant each regular connected component of each level-set of f and reverses its orientation.

Theorem 3. *Let D^2 be a 2-disk, $f\in\mathcal{F}(M)$. Suppose there exists $h\in\Delta^-(f)$, i.e. $\Delta^-(f)\neq\varnothing$. Then there exists another $g\in\Delta^-(f)$ such that $g=h$ in a neighborhood of ∂D and $g^2\in\mathcal{S}_{\mathrm{id}}(f)$.*

Theorem 4. *Let M be an orientable connected compact surface and $f\in\mathcal{F}(M)$. If $\Delta^-(f)\neq\varnothing$, then there exists another $g\in\Delta^-(f)$ such that $g^2\in\mathcal{S}_{\mathrm{id}}(f)$.*