On the group of isometries of foliated manifolds

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Let M be a connected Riemannian C^{∞}-manifold of dimension n. We will denote by (M, F) manifold M with k-dimensional foliation F on M.

Definition 1. If for the some C^r- diffeomorphism $\varphi : M \to M$ the image $\varphi(L_\alpha)$ of any leaf L_α of foliation F is a leaf of foliation F, we say that the φ is C^r- diffeomorphism of foliated manifold and write as $\varphi : (M, F) \to (M, F)$ [2].

Let's denote as $\mathrm{Diff}_F(M)$ the set of all C^r- diffeomorphisms of foliated manifold (M, F), where $r \geq 0$. The group $\mathrm{Diff}_F(M)$ is subgroup of $\mathrm{Diff}(M)$ and therefore it is topological group in compact open topology.

Recall a vector field X is called a foliated field if for every vector field Y, tangent to F, Lie brocket $[X, Y]$ also is tangent to F. It is known that flow of every foliated field consists of diffeomorphisms of foliated manifold (M, F) [1]. The set $L(M, F)$ of foliated vector fields is a Lie subalgebra of Lie algebra $V(M)$ [2]. It follows from here that the group $\mathrm{Diff}_F(M)$ contains the Lie group for which the Lie algebra is an algebra $L(M, F)$.

Let M be a smooth connected finite-dimensional Riemannian manifold.

Definition 2. An isometry $\varphi : M \to M$ is called an isometry of foliated manifold (M, F) if it is diffeomorphism of foliated manifold (M, F) [1].

We will denote by $\mathrm{Iso}_F(M)$ the set of all C^r-isometries of foliated manifold (M, F), where $r \geq 0$. We have that

$$\mathrm{Iso}_F(M) = \mathrm{Diff}_F(M) \bigcap \mathrm{Iso}(M).$$

Let us recall that vector field X on riemannian manifold (M, g) is called Killing field if its flow consists of isometries of Riemannian manifold (M, g), that is $L_X g = 0$, where g is riemannian metric, $L_X g$ denotes Lie derivative of the metric g with respect to X. If X is foliated Killing vector field, it's flow consists of isometries of foliated manifold (M, F).) The set $K(M, F)$ of foliated Killing vector fields is a Lie subalgebra of Lie algebra $L(M, F)$. It follows from here that the group $\mathrm{Iso}_F(M)$ contains the Lie group for which the Lie algebra is an algebra $K(M, F)$.

Theorem 3. *Let (M, F) be a foliated manifold where M is a smooth connected finite-dimensional Riemannian manifold. Then the group $\mathrm{Iso}_F(M)$ is closed subset of $\mathrm{Iso}(M)$ in compact open topology.*

Really Cartan's theorem states that on a closed subgroup of a Lie group there exists a differential structure with respect to which the closed subgroup is a Lie subgroup of a given Lie group.By using this fact we formulate following .

Theorem 4. *Let (M, F) be a foliated manifold where M is a smooth connected finite-dimensional Riemannian manifold. Then the group $\mathrm{Iso}_F(M)$ is Lie subgroup of Lie group $\mathrm{Iso}(M)$.*

References

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[2] I. Tamura. *Topology of Foliations: An Introduction*, American Mathematical Society. Providence, Rhode Island, 1992. http://bookre.org/reader?file=582002.