On quotient spaces and their spaces of continuous maps

Sergiy Maksymenko
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: maks@imath.kiev.ua

Eugene Polulyakh
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: polulyah@imath.kiev.ua

Let $p : X \to Y$ be a factor map between topological spaces, that is p is surjective and a subset $A \subset Y$ is open if and only if $p^{-1}(A)$ is open in X.

Let $\Delta = \{p^{-1}(y) \mid y \in Y\}$ be the partition of X into the inverse images of points of Y. A continuous map $h : X \to X$ will be called a Δ-*map* if for each $\omega \in \Delta$ its image $h(\omega)$ is contained in some element ω' of Δ. Hence every Δ-map h induces a map $\psi(h) : Y \to Y$ making commutative the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi(h)} & Y \end{array} \qquad (1)$$

It is well known that $\psi(h)$ is continuous whenever h is so.

Let $\mathcal{E}(X, \Delta)$ be the monoid of all Δ-maps of X, and $\mathcal{E}(Y) = C(Y, Y)$ be the monoid of all continuous self-maps of Y. Let also $\mathcal{H}(X, \Delta)$ be the subgroup of $\mathcal{E}(X, \Delta)$ consisting of homeomorphisms and $\mathcal{H}(Y)$ be the group of homeomorphisms of Y.

Then the correspondence $h \mapsto \psi(h)$ is a well defined map

$$\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y) \qquad (2)$$

being a homomorphism of monoids.

The following statement gives sufficient conditions under which ψ will be continuous with respect to compact open topologies on $\mathcal{E}(X, \Delta)$ and $\mathcal{E}(Y)$.

Lemma 1. *Let $p : X \to Y$ be a factor map having the following property:*

(K) for every compact subset $L \subset Y$ there exists a compact subset $K \subset X$ such that $p(K) = L$.

Then the homomorphism of monoids $\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y)$ is continuous with respect to compact open topologies.

Recall that a continuous map $p : X \to Y$

- is called *proper* if $p^{-1}(L)$ is compact for each compact $L \subset Y$;
- *admits local cross-sections* if for every $y \in Y$ there exists an open neighborhood V and a continuous map $f : V \to X$ such that $p \circ f = \mathrm{id}_V$.

Corollary 2. *Suppose Y is a locally compact Hausdorff space. Then each of the following conditions implies that the map $\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y)$ is continuous with respect to compact open topologies:*

(1) *p is a proper map;*
(2) *p is an open map and admits local cross sections;*
(3) *p is a locally trivial fibration.*

Let Y be a topological space. Say that two points $y, z \in Y$ are T_2-*disjoint* (in Y) if they have disjoint neighborhoods. Denote by $\mathrm{hcl}(y)$ the set of all $z \in Y$ that are *not T_2-disjoint from y*. Then

$z \in \mathrm{hcl}(y)$ if and only if each neighborhood of z intersects each neighborhood of y. We will call $\mathrm{hcl}(y)$ the *Hausdorff closure* of y.

We will say that $y \in Y$ is a *branch point* whenever $\mathrm{hcl}(y) \setminus y \neq \emptyset$, so there are points that are not T_2-disjoint from y. The set of all branch points of Y will be denoted by $\mathrm{Br}(Y)$.

Theorem 3. *Let X be a locally compact Hausdorff topological space, Y be a T_1-space whose set $\mathrm{Br}(Y)$ of branch points is locally finite, and $p : X \to Y$ be an open continuous and surjective map. Then for every compact $L \subset Y$ there exists a compact subset $K \subset X$ such that $p(K) = L$. In particular, due to Lemma 1, the map (2) $\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y)$ is continuous with respect to compact open topologies.*