On similarity of two families of matrices over a field

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Let $\mathbb{F}$ be a field of characteristic zero. Denote by $M_{m,n}(\mathbb{F})$ the set of $m \times n$ matrices over $\mathbb{F}$ and by $M_{m,n}(\mathbb{F}[\lambda])$ the set of $m \times n$ matrices over the polynomial ring $\mathbb{F}[\lambda]$.

In the ring $\mathbb{F}[\lambda]$ we consider the operation of differentiation $\mathbf{D}$. Let $a(\lambda) = \sum_{i=0}^{l} a_i \lambda^{l-i} \in \mathbb{F}[\lambda]$. Put $\mathbf{D}\,(a(\lambda)) = \sum_{i=0}^{l} (l-i) a_i \lambda^{l-i-1}$ and $\mathbf{D}^k(a(\lambda)) = \mathbf{D}(a^{(k-1)}(\lambda)) = a^{(k)}(\lambda)$ for every natural $k \geq 2$. The differentiation of a matrix $A(\lambda) = \left[\; a_{ij}(\lambda) \;\right] \in M_{m,n}(\mathbb{F}[\lambda])$ is understood as its elementwise differentiation, i.e., $A^{(1)}(\lambda) = \mathbf{D}(A(\lambda)) = [\mathbf{D}(a_{ij}(\lambda))] = [a_{ij}^{(1)}(\lambda)]$ and $A^{(k)}(\lambda) = \mathbf{D}(A^{(k-1)}(\lambda))$

Let $b(\lambda) = (\lambda - \beta_1)^{k_1}(\lambda - \beta_2)^{k_2} \cdots (\lambda - \beta_r)^{k_r} \in \mathbb{F}[\lambda]$, $\deg b(\lambda) = k = k_1 + k_2 + \cdots + k_r$, and $A(\lambda) \in M_{m,n}(\mathbb{F}[\lambda])$. For the monic polynomial $b(\lambda)$ and the matrix $A(\lambda)$ we define the matrix

$$M[A,b] = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_r \end{bmatrix} \in M_{mk,n}(\mathbb{F}), \quad \text{where } N_j = \begin{bmatrix} A(\beta_j) \\ A^{(1)}(\beta_j) \\ \vdots \\ A^{(k_j-1)}(\beta_j) \end{bmatrix} \in M_{mk_j,n}(\mathbb{F}),\ j = 1, 2, \ldots, r.$$

The Kronecker product of matrices $A = [a_{ij}]$ $(n \times m)$ and B is denoted by $A \otimes B = \left[\; a_{ij}B \;\right]$. Let nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be equivalent and $S(\lambda) = \operatorname{diag}(s_1(\lambda), \ldots, s_{n-1}(\lambda), s_n(\lambda))$ be their Smith normal form (see [5], Chapter 1). For $A(\lambda)$ and $B(\lambda)$ we define the matrix

$$D(\lambda) = \left(\Big(s_1(\lambda)s_2(\lambda)\cdots s_{n-1}(\lambda)\Big)^{-1} B^*(\lambda)\right) \otimes A^t(\lambda) \in M_{n^2,n^2}(\mathbb{F}[\lambda]),$$

where $A^t(\lambda)$ denote the transpose of $A(\lambda)$. It may be noted if $S(\lambda) = \operatorname{diag}(1, \ldots, 1, s(\lambda))$ is the Smith normal form of the matrices $A(\lambda)$ and $B(\lambda)$, then $D(\lambda) = B^*(\lambda) \otimes A^t(\lambda)$.

Definition 1. Two families of $n \times n$ matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_r\}$ over a field $\mathbb{F}$ are said to be similar if there exists a matrix $T \in GL(n, \mathbb{F})$ such that $A_i = TB_iT^{-1}$ for all $i = 1, 2, \ldots, r$.

The task of classifying square matrices up to similarity is one of the core and oldest problems in linear algebra (see [1]– [7] and references therein), and it is generally acknowledged that it is also one of the most hopeless problems already for $r = 2$. Standard approaches for deciding similarity depend upon the Jordan canonical form, the invariant factor algorithm and the Smith form, or the closely related rational canonical form. In numerical linear algebra, this leads to deep algorithmic problems, unsolved even up to this date, that are caused by numerical instabilities in solving eigenvalue problems or by the inability to effectively compute sizes of the Jordan blocks or degrees of invariant factors, if the matrix entries are not known precisely. At present such problems are called wild ([2], [3]).

The families $\mathbf{A}$ and $\mathbf{B}$ we associate with monic matrix polynomials

$$A(\lambda) = I_n\lambda^r + A_1\lambda^{r-1} + A_2\lambda^{r-2} + \cdots + A_r \quad \text{and} \quad B(\lambda) = I_n\lambda^r + B_1\lambda^{r-1} + B_2\lambda^{r-2} + \cdots + B_r$$

over a field $\mathbb{F}$ of degree r respectively, where I_n is the identity $n \times n$ matrix. It is clear that the families $\mathbf{A}$ and $\mathbf{B}$ are similar over $\mathbb{F}$ if and only if the matrices $A(\lambda)$ and $B(\lambda)$ are similar over $\mathbb{F}$. The purpose of this report is to give a criterion of similarity of two families of matrices over a field.

Theorem 2. *Let matrices* $A(\lambda) = I_n\lambda^r + \sum_{i=1}^{r} A_i\lambda^{r-i}, B(\lambda) = I_n\lambda^r + \sum_{i=1}^{r} B_i\lambda^{r-i} \in M_{n,n}(\mathbb{F}[\lambda])$ *of degree* r *be equivalent, and let* $S(\lambda) = \operatorname{diag}(s_1(\lambda), \dots, s_{n-1}(\lambda), s_n(\lambda))$ *be their Smith normal form. Further, let* $s_n(\lambda) = (\lambda-\alpha_1)^{k_1}(\lambda-\alpha_2)^{k_2}\cdots(\lambda-\alpha_r)^{k_r}$, *where* $\alpha_i \in \mathbb{F}$ *for all* $i = 1, 2, \dots, r$.

The families $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ *and* $\mathbf{B} = \{B_1, B_2, \dots, B_r\}$ *are similar over* $\mathbb{F}$ *if and only if* rank $M[D, s_n] < n^2$ *and the homogeneous system of equations* $M[D, s_n]x = \bar{0}$ *has a solution* $x = [v_1, v_2, \dots, v_{n^2}]^t$ *over* $\mathbb{F}$ *such that the matrix*

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ v_{n+1} & v_{n+2} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{n^2-n+1} & v_{n^2-n+2} & \dots & v_{n^2} \end{bmatrix} \in M_{n,n}(\mathbb{F})$$

is nonsingular. If $\det V \neq 0$, *then* $A_i = V^{-1}B_iV$ *for all* $i = 1, 2, \dots, r$.

Example 3. Let $\mathbb{F} = \mathbb{Q}$ be the field of rational numbers. Further, let $\mathbf{A} = \left\{A_1 = \begin{bmatrix} -3 & 0 \\ -4 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ and $\mathbf{B} = \left\{B_1 = \begin{bmatrix} 1 & 0 \\ -4 & -3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\right\}$ be two families of 2×2 matrices over the field $\mathbb{Q}$.

Monic matrix polynomials $A(\lambda) = I_2\lambda^2 + A_1\lambda + A_2 = \begin{bmatrix} \lambda^2 - 3\lambda + 1 & 1 \\ -4\lambda + 1 & \lambda^2 + \lambda + 1 \end{bmatrix}$ and $B(\lambda) = I_2\lambda^2 + B_1\lambda + B_2 = \begin{bmatrix} \lambda^2 + \lambda & 0 \\ -4\lambda + 1 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$ with entries from $\mathbb{Q}[\lambda]$ are equivalent and $S(\lambda) = \operatorname{diag}(1, (\lambda^2-1)(\lambda^2-2\lambda))$ is their Smith normal form. It may be noted that $s_1(\lambda) = 1$ and $s_2(\lambda) = \lambda(\lambda+1)(\lambda-1)(\lambda-2)$. Construct the matrix

$$D(\lambda) = B^*(\lambda) \otimes A^t(\lambda) = \begin{bmatrix} \lambda^2 - 3\lambda + 2 & 0 \\ 4\lambda - 1 & \lambda^2 + \lambda \end{bmatrix} \otimes \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -4\lambda + 1 \\ 1 & \lambda^2 + \lambda + 1 \end{bmatrix}$$

and solve the system of equations $M[D, s_2]x = \bar{0}$. Crossing out zero rows in the matrix $M[D, s_2]$ and after elementary transformations over the rows of this matrix we get the following system of linear equations

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 9 & 2 & 6 \\ 7 & 49 & 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this system of equations we obtain $x_1 = -x_2 = t$, $x_3 = 0$ and $x_4 = t$. It is obvious that the matrix $V = \begin{bmatrix} t & -t \\ 0 & t \end{bmatrix}$ is nonsingular for nonzero $t \in \mathbb{Q}$. Thus, the families of matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, i.e., $A_i = V^{-1}B_iV$, $i = 1, 2$.

References

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