Some connections between invariant factors of matrix and its submatrix

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Invariant factors and their connections play an important role in the studying of matrix's structure [3, 5]. For instance, at augmented one matrix with a single row to obtain another matrix are used the relationships between the invariant factors of these matrices. B.W. Jones [2] state a fact that a unimodular $m \times n$ $(m < n)$ matrix A over a principal ideal domain may always be augmented with a single row to obtain a unimodular $(m+1) \times n$ matrix B. Some relationships between the invariant factors of an arbitrary matrix A and those of a one row prolongation B over the same area was established by R. Thompson [4]. D. Carlson [1] obtained similar results in terms of a finitely generated module.

In this paper, we give necessary and sufficient conditions that a matrix A may be augmented with a single row to obtain a matrix B over elementary divisor domains.

Let R be an elementary divisor domain [4] with $1 \neq 0$, i.e., every $m \times n$ matrix A over R have diagonal reduction, namely $A \sim E = diag(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, k-1$, where the matrix E is called the Smith normal form, the diagonal elements ε_i are invariant factors of the matrix A. The notation $a|b$ means that the element a is the divisor of the element b, i.e., $b = ac$, where $c \in R$.

Theorem 1. *Let R be an elementary divisor domain, A be an $m \times n$ matrix over R, $A \sim \mathrm{E} = diag(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, k-1$. Let also $\delta_1, \ldots, \delta_k \in R$ be nonzero elements such that $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k-1$. Then the matrix A may be augmented with a single row to obtain an $(m+1) \times n$ matrix $B \sim \Delta = diag(\delta_1, \ldots, \delta_k, 0, \ldots, 0)$, $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k-1$, if and only if*

$$\delta_1 | \varepsilon_1 | \delta_2 | \varepsilon_2 | \ldots | \delta_k | \varepsilon_k.$$

References

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