A short note on Hurewicz and $\mathcal{I}$-Hurewicz properties in topological spaces

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Theorem 1. *Let X be an ϵ-space and let $\mathcal{I}$ be an ideal having a pseudounoin, then X satisfies $S_{fin}(\Omega, \mathcal{O}^{\mathcal{I}\text{-}gp})$ if and only if X has Hurewicz property* [4].

Lemma 2. [1, Theorem 4.1.2] *(see also [5]) An ideal $\mathcal{I}$ of $\mathbb{N}$ is meager if and only if there is a partition $\{P_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into finite sets such that each $A \in \mathcal{I}$ contains atmost finitely many $P_n' s$.*

Proposition 3. *For a space X and a meager ideal $\mathcal{I}$, X satisfies $S_{fin}(\Lambda, \mathcal{O}^{\mathcal{I}\text{-}gp})$ if and only if X has Hurewicz property.*

Problem 4. Is there a Lindelöf non-ϵ-space such that $S_{fin}(\Omega, \mathcal{O}^{\mathcal{I}\text{-}gp})$ holds but $S_{fin}(\Lambda, \mathcal{O}^{\mathcal{I}\text{-}gp})$ fails?.

Definition 5. A space X is said to have $\mathcal{I}$-Hurewicz property (in short $\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}, \mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \cup \mathcal{V}_n\} \in \mathcal{I}$[2].

Theorem 6. *Let X be an ϵ-space satisfying $CDR_{sub}(\Lambda, \Lambda)$ and let $\mathcal{I}$ be a meager ideal of $\mathbb{N}$. If X has $\mathcal{I}$-Hurewicz property then X also has Hurewicz property.*

Theorem 7. *If a filter $\{$ does not have $\mathcal{I}$-Hurewicz property then $\chi(\mathcal{F}) \geq \mathfrak{b}(\mathcal{I})$.*

Remark 8. CH denotes the Continuum Hypothesis. Assume $\neg CH$. Let $\mathcal{I}$ be an ideal of $\mathbb{N}$ and let k be an infinite cardinal satisfying $\mathfrak{b} < k < \mathfrak{b}(\mathcal{I})$. There is $X \subset \mathbb{N}^{\mathbb{N}}$ of size $\mathfrak{b}$ which is not a Hurewicz space. But X is $\mathcal{I}$-Hurewicz.

Example 9. There exists a non-$\mathcal{I}$-Hurewicz filter of character $\mathfrak{b}(\mathcal{I})$. Consider a set $\{f_\alpha : \alpha < \mathfrak{b}(\mathcal{I})\}$ which is not $\mathcal{I}$-bounded. Let $\mathcal{F}$ be a filter on $\mathbb{N} \times \mathbb{N}$ generated by the family $\{F_\alpha : \alpha < b(\mathcal{I})\}$ where $F_\alpha = \{(n, m) : m \geq f_\alpha(n), n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, $\mathcal{U} = \{U(n, m) : m \in \mathbb{N}\}$ is an open cover of $\mathcal{F}$ where for each $n, m \in \mathbb{N}$, $U(n, m) = \{A \subset \mathbb{N} \times \mathbb{N} = \min\{k \in \mathbb{N} : (n, k) \in A\}\}$.

References

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