Asymptotically best possible Lebesgue inequalities on the classes of generalized Poisson integrals

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Denote by $C^{\alpha,r}_{\beta}C,\ \alpha>0,\ r>0$, (see, e.g., [1]) the set of all 2π–periodic functions, sucht that for all $x\in\mathbb{R}$ can be represented in the form of convolution

$$f(x)=\frac{a_0}{2}+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}P_{\alpha,r,\beta}(x-t)\varphi(t)dt,\ a_0\in\mathbb{R},\ \varphi\perp 1, \tag{1}$$

where $\varphi\in C$, and $P_{\alpha,r,\beta}(t)$ is a generalized Poisson kernel of the form

$$P_{\alpha,r,\beta}(t)=\sum_{k=1}^{\infty}e^{-\alpha k^r}\cos\big(kt-\frac{\beta\pi}{2}\big),\ \alpha>0,\ r>0,\ \beta\in\mathbb{R}.$$

If f and φ are connected with a help of equality (1), then the function f in this equality is called the generalized Poisson integral of the function φ and is denoted by $J^{\alpha,r}_{\beta}(\varphi)$. The function φ in the equality (1) is called the generalized derivative of the function f and is denoted by $f^{\alpha,r}_{\beta}$.

By $\rho_n(f;x)$ we denote the deviation of the function f from its partial Fourier sum of order $n-1$:

$$\rho_n(f;x):=f(x)-S_{n-1}(f;x),$$

where

$$S_{n-1}(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n-1}\left(a_k\cos kx+b_k\sin kx\right),$$

$$a_k=a_k(f)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(t)\cos ktdt,\quad b_k=b_k(f)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(t)\sin ktdt,$$

and by $E_n(f)_C$ we denote the best uniform approximation of the function f by elements of the subspace τ_{2n-1} of trigonometric polynomials $t_{n-1}(\cdot)$ of the order $n-1$:

$$E_n(f)_C:=\inf_{t_{n-1}\in\tau_{2n-1}}\|f-S_{n-1}(f)\|_C.$$

The norms $\|\rho_n(f;\cdot)\|_C$ can be estimated via $E_n(f)_C$, using the Lebesgue inequality

$$\|\rho_n(f;\cdot)\|_C\leq\left(\frac{4}{\pi^2}\ln n+\mathcal{O}(1)\right)E_n(f)_C,\ n\in\mathbb{N}. \tag{2}$$

On the whole space C the inequality (2) is asymptotically exact. At the same time for the sets of functions $C^{\alpha,r}_{\beta}C$ the inequality (2) is not asymptotically exact.

We establish the asymptotically best possible Lebesgue-type inequalities for the functions $f\in C^{\alpha,r}_{\beta}C$, in which for all n, starting from the number $n_1=n_1(\alpha,r)$, an additional term is estimated by absolute constant.

For arbitrary $\alpha > 0$, $r \in (0,1)$ we denote by $n_1 = n_1(\alpha, r)$ the smallest integer $n \in \mathbb{N}$, such that

$$\frac{1}{\alpha r}\frac{1}{n^r}\left(1 + \ln\frac{\pi n^{1-r}}{\alpha r}\right) + \frac{\alpha r}{n^{1-r}} \leq \frac{1}{(3\pi)^3}. \tag{3}$$

Theorem 1. *Let $\alpha > 0$, $r \in (0,1)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, for any function $f \in C^{\alpha,r}_{\beta}C$ and all $n \geq n_1(\alpha, r)$ the following inequality holds*

$$\|\rho_n(f;\cdot)\|_C \leq e^{-\alpha n^r}\left(\frac{4}{\pi^2}\ln\frac{n^{1-r}}{\alpha r} + \gamma_n\right)E_n(f^{\alpha,r}_{\beta})_C. \tag{4}$$

Moreover, for arbitrary function $f \in C^{\alpha,r}_{\beta}C$ one can find a function $F(x) = F(f,n,x)$ from the set $C^{\alpha,r}_{\beta}C$, such that $E_n(F^{\alpha,r}_{\beta})_C = E_n(f^{\alpha,r}_{\beta})_C$, such that for $n \geq n_1(\alpha, r)$ the equality holds

$$\|\rho_n(F;\cdot)\|_C = e^{-\alpha n^r}\left(\frac{4}{\pi^2}\ln\frac{n^{1-r}}{\alpha r} + \gamma_n\right)E_n(f^{\alpha,r}_{\beta})_C. \tag{5}$$

In (4) and (5) for the quantity $\gamma_n = \gamma_n(\alpha, r, \beta)$ the estimate holds $|\gamma_n| \leq 20\pi^4$.

References

[1] A.I. Stepanets Methods of Approximation Theory. VSP: Leiden, Boston, 2005.