Differential invariants of transformations group

Abdigappar Narmanov
(National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan)
E-mail: narmanov@yandex.ru

Xurshid Sharipov
(National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan)
E-mail: sh_xurshid@yahoo.com

Definition 1. Let M, B be smooth manifolds and $p \in M$. Let $f, g : M \to B$ be smooth mappings satisfying the condition $f(p) = g(p) = q$.

1) f has a first-order tangency with g at the point p if $(df)_p = (dg)_p$ as the map $T_pM \to T_pB$.

2) f has a contact of k th order with g at the point p if the map $(df) : TM \to TB$ has a contact of order $(k-1)$ with map (dg) at each point T_pM. This fact can be written as follows: $f \sim_k g$ at the point p (k -positive number) [2].

We denote by $J^k(M,B)_{p,q}$ the sets of equivalence classes with respect to the "$\sim_k$ at the point p" in the space of mappings $f : M \to B$ satisfying the condition $f(p) = q$. We put $J^k(M,B) = \bigcup_{(p,q)\in M\times B} J^k(M,B)_{p,q}$.

Definition 2. The set $J^k(M,B)$ is called the space of k -jets.

The action of the group G on M gives rise to some action of the group on $J^k(M,B)$. This action is called the k -prolongation of the group G on $J^k(M,B)$.

We let $G^{(k)}$ denote the associated prolonged group action on the jet space $J^k(M,B)$. The infinitesimal generators of the k -th prolongation of the group G to $J^k(M,B)$ are k - prolongations of infinitesimal generators of the group G.

Definition 3. The function $I \in C^\infty(J^k(M,B))$ is called a differential invariant of order k of the group G if it is preserved under the action of the k -th prolongation G on $J^k(M,B)$, that is, $g(I) = I$ for any transformation $g \in G^{(k)}$.

Differential invariants of Lie group of transformations are studied in the papers [1], [3], [4].

Let G be a Lie group of transformations of the space of two independent u, v and three dependent x_1, x_2, x_3 variables, and following vector field

$$X = \xi_1(u,v,x)\frac{\partial}{\partial u} + \xi_2(u,v,x)\frac{\partial}{\partial v} + \sum_{i=1}^{3} \eta_i(u,v,x)\frac{\partial}{\partial x_i} \tag{1}$$

is infinitesimal generator of the group G.

It is known that any Lie group is similar to the group of translations. This property of the groups is remarkable and its use permits simplification of finding of differential invariants of the group.

In order to use this possibility we produce the replacement of variables.

Let us consider functions $F_1(u,x)$ and $F_2(v,x)$ which are solutions of following equation

$$X(F) = 1. \tag{2}$$

Let $I_1(u,v,x)$, $I_2(u,v,x)$ and $I_2(u,v,x)$ be are functionally independent invariant functions of the group G, i.e. they are satisfy following equations

$$X(I_i) = 0, i = 1,2,3. \tag{3}$$

We will replace the variables in the space of (u, v, x_1, x_3, x_3) by putting

$$s = F_1(u, x), t = F_2(v, x), \tag{4}$$

$$y_i = I_i(u, v, x), \tag{5}$$

where $i = 1, 2, 3$. Using easy deductions, we can verify that in variables (s, t, y_1, y_2, y_3) the vector field (1) has the following form

$$X = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}. \tag{6}$$

This form of the vector field X shows that the group G is similar to the group of translations. Moreover in the coordinates (s, t, y_1, y_2, y_3) for any $k \in N$ for $k - th$ prolongation $X^{(k)}$ of the vector field (6) it holds equality $X^{(k)} = X$.

Let us recall differentiation operator D is called invariant differentiation operator with respect group G if it holds $DX(F) = XD(F)$ for any smooth function F.

It follows from the form of the vector field (6) invariant differentiation operators for the group G are following operators of total derivatives: $D = D_s + D_t$.

If we put

$$p_{i,k} = \frac{\partial^k y_i}{\partial s^k}, q_{i,k} = \frac{\partial^k y_i}{\partial t^k}$$

then we can write total derivatives in following forms:

$$D_s = \frac{\partial}{\partial s} + \sum_{i=1}^{3} p_{i,1} \frac{\partial}{\partial y_i} + \sum_{i=1}^{3} p_{i,2} \frac{\partial}{\partial p_{i,1}} + ... \tag{7}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{i=1}^{3} q_{i,1} \frac{\partial}{\partial y_i} + \sum_{i=1}^{3} q_{i,2} \frac{\partial}{\partial q_{i,1}} + ... \tag{8}$$

We have the following equalities

$$D_s = \frac{1}{D_u F_1} D_u, D_t = \frac{1}{D_v F_2} D_v, \tag{9}$$

which will allow us to return to the old variables, where D_u, D_v- also operators of total derivatives with respect u, v.

Let denote by $D^k(F)$ derivatives $D_s^k + D_t^k(F)$ of order k.

Thus we have the following theorem.

Theorem 4. *Suppose I_1, I_2, I_3 are independent invariants of the group G, $F_1(u, x)$, $F_2(v, x)$, are solutions of the equation $X(F) = 1$. Then functions $I_i(u, v, x)$ and $D^k(I_i)$ are differential invariants of order k.*

References

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