Non-acyclic SL_2-representations of twist knots and non-trivial L-invariants

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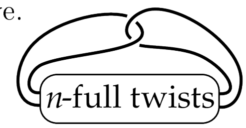
In this talk on the joint work [TTU20] with **Ryoto Tange** in Kogakuin university and **Anh T. Tran** in the University of Texas at Dallas, we study irreducible SL_2-representations of twist knots. For each $n \in \mathbb{Z}$, *the twist knot* $J(2,2n)$ is defined by the diagram below, the horizontal twists being right handed if n is positive and left handed if negative.



We have $J(2,0) = 0_1$ (unknot), $J(2,2) = 3_1$ (trefoil), $J(2,4) = 5_2$, and $J(2,-2) = 4_1$ (figure-eight knot). Regarding a 1/2-full twist to be a half twist, $J(2,-2n)$ and $J(2,2n+1)$ are the mirror images to each other, hence we only consider $J(2,2n)$. The knot group $\pi_n := \pi_1(S^3 - J(2,2n))$ of $J(2,2n)$ admits the presentation

$$\pi_n = \langle a, b \,|\, aw^n = w^n b \rangle\,, \quad w = [a, b^{-1}] = ab^{-1}a^{-1}b.$$

by [HS04, Proposition 1]. Since twist knots are 2-bridge knots, the Culler–Shalen theory of character variety together with Riley's calculation assures that conjugacy classes of $\rho \in \mathrm{Hom}(\pi_n, \mathrm{SL}_2(\mathrm{SL}))$ are parametrized by $x := \mathrm{tr}\,\rho(a)$ and $y := \mathrm{tr}\,\rho(ab)$. A representation ρ is said to be *acyclic* if $H_i(\pi, \rho) = 0$ holds for every i and *non-acyclic* if otherwise. Here is our first theorem.

Theorem 1. *Conjugacy classes of non-acyclic irreducible $\mathrm{SL}_2(\mathrm{SL})$-representations of $J(2,2n)$ are exactly given by $x = y = 1 - 2\cos\dfrac{2\pi k}{3n-1}$, $0 < k \leq \dfrac{|3n-1|-1}{2}$, $k \in \mathbb{Z}$.*

This implies that every such representation corresponds to a point on the diagonal $x = y$ in $\mathbb{R}^2 \subset \mathrm{SL}^2$. In order to prove this assertion, we investigate the intersection of curves defined by Chebyshev-like polynomials $f_n(x,y)$, $\tau_n(x,y) \in \mathbb{Z}[x,y]$. The polynomial $f_n(x,y)$ defines a component of the character variety and coincides with the Riley polynomial $\Phi_n(x,u)$ via $-u = y - x^2 + 2$. The polynomial $\tau_n(x,y)$ is the Reidemeister torsion regarded as a function so that $\tau_n(x,y) = 0$ iff a representation ρ with $(\mathrm{tr}\,\rho(a), \mathrm{tr}\,\rho(ab)) = (x,y)$ is non-acyclic. We first prove that the intersection of their zeros lie on $x = y$ and then determine all common roots of $f_n(x,x)$ and $\tau_n(x,x)$. We also introduce several Chebyshev-like polynomials g_n, h_n, $k_n \in \mathbb{Z}[x]$ and prove $f_n(x,x) = g_n k_n$, $\tau_n(x,x) = h_n k_n$, where k_n is the greatest common divisor. We in addition prove the following theorem, generalizing [Bén20, Remark 4.6].

Theorem 2. *The two curves $f_n(x,y) = 0$ and $\tau_n(x,y) = 0$ in $\mathbb{R}^2$ have a common tangent line at every intersection point, while the second derivatives of their implicit functions do not coincide. In other words, every zero of $\tau_n(x,y)$ on $f_n(x,y) = 0$ has multiplicity two in the function ring $\mathrm{SL}[x,y]/(f_n(x,y))$.*

The following theorems characterize non-acyclic representations.

Theorem 3. *The conjugacy class of an irreducible $\mathrm{SL}_2(\mathrm{SL})$-representation ρ of $J(2,2n)$ is on the line $x = y$ if and only if ρ factors through the -3-Dehn surgery.*

Theorem 4. *The conjugacy class of an irreducible $\mathrm{SL}_2(\mathrm{SL})$-representation ρ of $J(2,2n)$ on $x = y$ is non-acyclic if and only if $\rho(a^{-1}w^n)$ is of order 3.*

Our study is indeed motivated by a problem in arithmetic topology. We finally investigate the L-invariants of universal deformations of residual representations, which was introduced in [KMTT18] in a perspective of the Hida–Mazur theory. Let $\overline{\rho} : \pi_n \to \mathrm{SL}_2(F)$ be a representation over a field F with char $= p > 2$ and a completed discrete valuation ring (CDVR) O with the residue field F. A *deformation* (or a *lift*) of $\overline{\rho}$ over a complete local O-algebra R is a representation $\rho : \pi_n \to \mathrm{SL}_2(R)$ with the residual representation $\overline{\rho}$. A *universal deformation* $\boldsymbol{\rho} : \pi_n \to \mathrm{SL}_2(\mathcal{R})$ of $\overline{\rho}$ over O is a deformation such that any deformation over any R uniquely factors through $\boldsymbol{\rho}$ up to strict equivalence. If $\boldsymbol{\rho}$ is absolutely irreducible, then $\boldsymbol{\rho}$ uniquely exists up to O-isomorphism and strict equivalence.

When $\mathcal{R}$ is a Noetherian UFD and the group homology $H_1(\pi_n, \boldsymbol{\rho})$ with local coefficients is a finitely generated torsion $\mathcal{R}$-module, the L-invariant $L_{\boldsymbol{\rho}} \in \mathcal{R}/\doteq$ is defined to be the order of $H_1(\pi_n, \boldsymbol{\rho})$, where $\doteq$ denotes the equality up to multiplication by units in $\mathcal{R}$. Let $\Delta_{\boldsymbol{\rho},i}(t)$ denote the i-th $\boldsymbol{\rho}$-twisted Alexander polynomials. Then we have $L_{\boldsymbol{\rho}} \doteq \Delta_{\boldsymbol{\rho},1}(1)$. A general theory of twisted invariants yields $L_{\boldsymbol{\rho}} \doteq \tau_{\boldsymbol{\rho}}\Delta_{\boldsymbol{\rho},0}(0)$. For most cases we have $\Delta_{\boldsymbol{\rho},0} \doteq 1$, so that we have $L_{\boldsymbol{\rho}} \not\doteq 1$ if and only if $\tau_{\overline{\rho}} = 0$, that is, $\overline{\rho}$ is non-acyclic. Now B. Mazur's Question 2 in [Maz00, page 440] may be varied as follows:

Problem 5. *Investigate the L-invariants $L_{\boldsymbol{\rho}}$ of the universal deformations $\boldsymbol{\rho}$ over O of absolutely irreducible non-acyclic residual representations $\overline{\rho}$.*

The following theorem completely answers to this problem, that is, it determines all residual representations with non-trivial L-invariants, as well as explicitly determine the L-invariants themselves.

Theorem 6. Every absolutely irreducible representation $\overline{\rho} : \pi_n \to \mathrm{SL}_2(F)$ of a twist knot corresponds to a root of k_n in F. Suppose that $\overline{\rho}$ corresponds to a root $\overline{\alpha}$ of k_n with multiplicity m and that $\alpha_1 = \alpha, \cdots, \alpha_m$ are distinct lifts of $\overline{\alpha}$ with $k_n(\alpha_i) = 0$ and $\alpha \in O$. If $\dfrac{\partial f_n}{\partial y}(\overline{\alpha}, \overline{\alpha}) \neq 0$ holds, so that there is a universal deformation $\boldsymbol{\rho} : \pi_n \to \mathrm{SL}_2(O[[x - \alpha]])$ over O, then the equalities

$$L_{\boldsymbol{\rho}} \doteq k_n(x)^2 \doteq \prod_i (x - \alpha_i)^2$$

in $\mathcal{R} = O[[x - \alpha]]$ hold. If in addition $p \nmid 3n - 1$, then $m = 1$ and $L_{\boldsymbol{\rho}} \doteq (x - \alpha)^2$ holds.

If instead $\dfrac{\partial f_n}{\partial x}(\overline{\alpha}, \overline{\alpha}) \neq 0$, then a similar equality holds in $\mathcal{R} = O[[y - \alpha]]$.

We remark that our work is derived from the scope of the following dictionary of analogy between knots and prime numbers (cf. [MT07, MTTU17, KMTT18], [Mor12, Chapter 14]).

Low dimensional topology	Number theory
Deformation space of hyperbolic structures	Universal p-ordinary modular deformation space
Dehn surgery points with $\mathbb{Z}$-coefficient	Arithmetic points

References

[Bén20] Léo Bénard, *Torsion function on character varieties*, to appear in Osaka J. Math. arXiv:1711.08781v2, 2020.

[HS04] Jim Hoste and Patrick D. Shanahan, *A formula for the A-polynomial of twist knots*, J. Knot Theory Ramifications **13** (2004), no. 2, 193–209. MR 2047468

[KMTT18] Takahiro Kitayama, Masanori Morishita, Ryoto Tange, and Yuji Terashima, *On certain L-functions for deformations of knot group representations*, Trans. Amer. Math. Soc. **370** (2018), 3171–3195.

[Maz00] Barry Mazur, *The theme of p-adic variation*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 433–459. MR 1754790 (2001i:11064)

[Mor12] Masanori Morishita, *Knots and primes*, Universitext, Springer, London, 2012, An introduction to arithmetic topology. MR 2905431

[MT07] Masanori Morishita and Yuji Terashima, *Arithmetic topology after Hida theory*, Intelligence of low dimensional topology 2006, Ser. Knots Everything, vol. 40, World Sci. Publ., Hackensack, NJ, 2007, pp. 213–222. MR 2371728

[MTTU17] Masanori Morishita, Yu Takakura, Yuji Terashima, and Jun Ueki, *On the universal deformations for* SL_2*-representations of knot groups*, Tohoku Math. J. (2) **69** (2017), no. 1, 67–84. MR 3640015

[TTU20] Ryoto Tange, Anh T. Tran, and Jun Ueki, *Non-acyclic* SL_2*-representations of twist knots*, preprint.