Functors and fuzzy metric spaces

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H. Toruńczyk and J. West [11] considered the construction of (completed) infinite iteration of the hyperspace functor and established some of its geometric properties. A counterpart of this construction for the superextension functor was investigated in [12]. Later, V. Fedorchuk [3] introduced a general notion of perfectly metrizable functor and obtained generalizations of results from [12].

Having in mind increasing interest to the fuzzy metric spaces we are going to extend the notion of perfectly metrizable functor over the class of fuzzy metric spaces in the sense of George and Veeramani [5].

Let X be a set, $*\colon [0,1]\times[0,1]\to[0,1]$ be a continuous t-norm. A GV-fuzzy metric on X is a mapping $m\colon X\times X\times\mathbb{R}_+\to(0,1]$ satisfies the following conditions for all $x,y,z\in X$, $s,t\in\mathbb{R}_+$:

(1GV) $m(x,y,t)>0$;
(2GV) $m(x,y,t)=1$ if and only if $x=y$;
(3GV) $m(x,y,t)=m(y,x,t)$;
(4GV) $m(x,z,t+s)\geq m(x,y,t)*m(y,z,s)$;
(5GV) the function $m(x,y,-)\colon\mathbb{R}_+\to[0,1]$ is continuous.

We will consider the class of (closed to) normal functors in the category **Comp** of compact Hausdorff spaces and continuous maps. For any such a functor F, there exists a natural transformation $\eta\colon 1_{\mathbf{Comp}}\to F$.

Suppose that to each compact fuzzy metric space $(X,m_X,*)$ a fuzzy metric $m_{F(X)}$ on the space $F(X)$ is assigned (with respect to the same triangular norm $*$). We will make the following assumptions:

(1) If $f\colon(X,m_X)\to(Y,m_Y)$ is an isometric embedding, then so is

$$F(f)\colon(F(X),m_{F(X)})\to(F(Y),m_{F(Y)}).$$

(2) The map $\eta_X\colon(X,m_X)\to(F(X),m_{F(X)})$ is an isometric embedding.
(3) $\mathrm{diam}(X,m_X)=\mathrm{diam}(F(X),m_{F(X)})$.

We suppose now that the mentioned functor F is a functorial part of a monad (F,η,ψ). Then we additionally require that the following holds.

(4) The map $\psi_X\colon(F^2(X),m_{F^2(X)})\to(F(X),m_{F(X)})$ is nonexpanding.

The condition in the definition from [3] that concerns the preservation of uniform continuity (this is equivalent to the preservation of (ε,δ)-continuity) does not have a unique counterpart in the case of fuzzy metric spaces, as in the latter case there are different notions of uniform continuity (see, e.g., [6, 4]).

In the talk, we discuss the question of completion of the metric direct limits of the form

$$F^+(X)=\varinjlim\{X\to F(X)\to F^2(X)\to\dots\},$$

with bonding maps $F^n(X)\to F^{n+1}(X)$ taken from the set $\{F^j(\eta_{F^{n-j}(X)})\mid 0\leq j\leq n\}$. Remark that the completion of fuzzy metric spaces does not necessarily exist [7].

Next, we consider the question of embedding of the (completions of the) spaces $F^+(X)$ in the spaces of the form

$$F^\omega(X) = \varprojlim\{F(X) \leftarrow F^2(X) \leftarrow F^3(X) \leftarrow \dots\}$$

As an example, we consider the hyperspace functor ([8]; see also [9]). Another example is the functor of idempotent measures; its fuzzy metrization is constructed in [2]. Recall that the idempotent measures are counterparts of the probability measures in idempotent mathematics, i.e., the part of mathematics in which at least one of arithmetic operations in $\mathbb{R}$ is replaced by an idempotent operation (e.g., max or min).

In the paper [10], a metrization of functors of finite degree is constructed. The considerations of this paper are significantly extended in [1], where the so called ℓ^p-metrics are defined on the sets of the form $F(X)$, where F is a functor with finite supports on the category of sets.

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