ON \textit{m-convexity and m-semiconvexity of sets in Euclidean spaces}

Tetiana Osipchuk

(Institute of Mathematics NAS of Ukraine, Kyiv)

E-mail: osipchuk@imath.kiev.ua

The topological and geometric properties of classes of generally convex sets in multidimensional real Euclidean space $\mathbb{R}^n$, $n \geq 2$, known as \textit{m-convex}, weakly \textit{m-convex}, \textit{m-semiconvex}, and weakly \textit{m-semiconvex}, $m = 1, 2, \ldots, n - 1$, are studied in [1]–[6]. A set of the space $\mathbb{R}^n$ is called \textit{m-convex} (\textit{m-semiconvex}) if for any point of the complement of the set to the whole space there is an $m$-dimensional plane (half-plane) passing through this point and not intersecting the set. An open set of the space is called \textit{weakly m-convex} (\textit{weakly m-semiconvex}), if for any point of the boundary of the set there exists an $m$-dimensional plane (half-plane) passing through this point and not intersecting the given set. A closed set of the space is called \textit{weakly m-convex} (\textit{weakly m-semiconvex}) if it is approximated from the outside by a family of open weakly \textit{m-convex} (weakly \textit{m-semiconvex}) sets. These notions were proposed by Professor Yuri Zelinskii [1], [2].

Let us denote the classes of \textit{m-convex} and weakly \textit{m-convex} sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathcal{C}_m^n$ and $\mathcal{W}_m^n$, respectively. There are weakly \textit{m-convex} sets in $\mathbb{R}^n$, $n \geq 2$, $1 \leq m < n$, which are not \textit{m-convex}, i.e., the class $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$ is not empty for any $m = 1, 2, \ldots, n - 1$. The example of an open set of the class $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$ is constructed in [4]. The examples of open and closed sets of $\mathcal{W}_m^{n-1} \setminus \mathcal{C}_m^{n-1}$ and examples of open sets of $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$, $n \geq 3$, $1 \leq m < n - 1$, are constructed in [6]. Moreover, any open or compact set of $\mathcal{W}_m^{n-1} \setminus \mathcal{C}_m^{n-1}$ is necessarily disconnected, but there exist domains of $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$, $n \geq 3$, $1 \leq m < n - 1$, which show the following three theorems.

**Theorem 1.** ([4]) An open set of the class $\mathcal{W}_m^{n-1} \setminus \mathcal{C}_m^{n-1}$ consists of at least three connected components.

**Theorem 2.** ([6]) A compact set of the class $\mathcal{W}_m^{n-1} \setminus \mathcal{C}_m^{n-1}$ consists of at least three connected components.

**Theorem 3.** ([6]) There exist domains of the class $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$, $n \geq 3$, $1 \leq m < n - 1$.

It is also known the topological classification of open (weakly) $(n - 1)$-convex sets in the space $\mathbb{R}^n$ with smooth boundary [1], [4]. Each such a set is convex, or consists of no more than two unbounded connected components, or is given by the Cartesian product $E^1 \times \mathbb{R}^{n-1}$, where $E^1$ is a subset of $\mathbb{R}$.

Let us denote the classes of \textit{m-semiconvex} and weakly \textit{m-semiconvex} sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathcal{S}_m^n$ and $\mathcal{W}_m^n$, respectively. In [3] it is constructed an example of an open set of the class $\mathcal{W}_m^n \setminus \mathcal{S}_m^n$. It is also conjectured that any open set of $\mathcal{W}_m^n \setminus \mathcal{S}_m^n$ consists of at least three components. The latter statement is proved in [4]. There can be also constructed sets of $\mathcal{W}_m^n \setminus \mathcal{S}_m^n$ and the example of domains of $\mathcal{W}_m^n \setminus \mathcal{S}_m^n$, $n \geq 3$, $1 \leq m < n - 1$, similar to the domains of $\mathcal{W}_m^n \setminus \mathcal{C}_m^n$. The following theorem shows the impossibility of the topological classification of weakly $1$-semiconvex sets with smooth boundary similar to the topological classification of open $(n - 1)$-convex and weakly $(n - 1)$-convex sets with smooth boundary.

**Theorem 4.** ([5]) An open, bounded set of the class $\mathcal{W}_m^n \setminus \mathcal{S}_m^n$ with smooth boundary consists of at least four connected components.

**References**


