Morita equivalence of non-commutative Noetherian schemes

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This is a joint work with Igor Burban, see [1].

The classical Morita theorem (see, for instance, [3, Ch. 18]) claims that the categories of modules over rings A and B are equivalent if and only if there is a finitely generated projective generator P of the category of right A-modules such that $\operatorname{End}_A P \simeq B$. Then this equivalence is established by the functor $P \otimes_A -$. If A and B are Noetherian, the same is the criterion of equivalence of their categories of finitely generated modules. On the other hand, Gabriel [2] proved that two Noetherian schemes X and Y are isomorphic if and only if the categories of coherent (or, which is the same, of quasi-coherent) sheaves of $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules are equivalent. We present here a result which is, in some sense, a combination and generalization of these two classical theorems.

Definition 1. (1) A *non-commutative Noetherian scheme* (*NCNS*) is a pair $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$, where X is a separated Noetherian scheme and $\mathcal{O}_{\mathbb{X}}$ is a sheaf of $\mathcal{O}_X$-algebras which is coherent as a sheaf of $\mathcal{O}_X$-modules. We denote by $\operatorname{Coh}\mathbb{X}$ and $\operatorname{QCoh}\mathbb{X}$ respectively the categories of coherent and quasi-coherent sheaves of left $\mathcal{O}_{\mathbb{X}}$-modules.

Note that the category $\operatorname{QCoh}\mathbb{X}$ is locally Noetherian and $\operatorname{Coh}\mathbb{X}$ is its subcategory of Noetherian objects. Therefore, they uniquely define each other.

(2) Two NCNS $\mathbb{X}$ and $\mathbb{Y}$ are called *Morita equivalent* if the categories $\operatorname{Coh}\mathbb{X}$ and $\operatorname{Coh}\mathbb{Y}$ (or, which is the same, $\operatorname{QCoh}\mathbb{X}$ and $\operatorname{QCoh}\mathbb{Y}$) are equivalent.

(3) A NCNS $\mathbb{X}$ is called *central* if $\mathcal{O}_X$ coincides with the center of $\mathcal{O}_{\mathbb{X}}$, i.e. for every point $x \in X$ the ring $\mathcal{O}_{X,x}$ is the center of the algebra $\mathcal{O}_{\mathbb{X},x}$.

Proposition 2. *For every NCNS $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$ there is a Noetherian scheme Z and a morphism $\phi : Z \to X$ such that the NCNS $\tilde{\mathbb{X}} = (Z, \phi^*\mathcal{O}_{\mathbb{X}})$ is central and Morita equivalent to $\mathbb{X}$. Moreover, the ring of global sections $\Gamma(Z, \mathcal{O}_Z)$ is isomorphic to the* center *of the category* $\operatorname{Coh}\mathbb{X}$*, i.e. the endomorphism ring of the identity functor* $\mathrm{id}_{\operatorname{Coh}\mathbb{X}}$*. If the scheme X is excellent, the morphism ϕ is finite.*

Thus, studying Morita equivalence, we can only consider central schemes. The following result is an analogue of the Gabriel's theorem.

Theorem 3. *If a NCNS $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$ is central, the scheme X is determined by the category* $\operatorname{QCoh}\mathbb{X}$ *(or, which is the same, by* $\operatorname{Coh}\mathbb{X}$*) up to an isomorphism.*

Actually, we give an explicit construction that restores X from $\operatorname{QCoh}\mathbb{X}$, namely, from the so called *spectrum* of this category in the sense of Gabriel [2], i.e. isomorphism classes of indecomposable injective objects. It is important that this construction also recovers affine open coverings of X.

Definition 4. A coherent sheaf of right $\mathcal{O}_{\mathbb{X}}$-modules $\mathcal{P}$ is called a *local progenerator* for $\mathbb{X}$ if for every point $x \in X$ its stalk $\mathcal{P}_x$ is a projective generator of the category of right $\mathcal{O}_{\mathbb{X},x}$-modules.

Our main result if the following.

Theorem 5. *Let $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$ and $\mathbb{Y} = (Y, \mathcal{O}_{\mathbb{Y}})$ be central NCNS. They are Morita equivalent if and only if there is an isomorphism $\phi : Y \to X$ and a local progenerator $\mathcal{P}$ for $\mathbb{X}$ such that $\phi^*(\mathcal{E}nd_{\mathcal{O}_{\mathbb{X}}}\mathcal{P}) \simeq \mathcal{O}_{\mathbb{Y}}$. Then this equivalence is established by the functor $\phi^*(\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{X}}} -)$.*

Note that even if $X = Y$, the isomorphism ϕ need not be identity. If it is so, this equivalence is called *central*.

We also specialize this theorem for the case of *non-commutative curves*, where it gives a sort of "globalization" of the known results on the local–global correspondence from the theory of lattices over orders (or integral representations of rings).

Definition 6. A *non-commutative curve* is a NCNS $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$ such that X is excellent and of pure dimension 1 and $\mathcal{O}_{\mathbb{X}}$ is *reduced*, i.e. contains no nilpotent ideals.

We always suppose $\mathbb{X}$ *central* and *connected* (in the central case, it just means that X is connected). We denote by $\mathcal{Q}_X$ the sheaf of fractions of $\mathcal{O}_X$ and set $\mathcal{Q}_{\mathbb{X}} = \mathcal{Q}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{X}}$. We denote $Q(X) = \Gamma(X, \mathcal{Q}_X)$ and $Q(\mathbb{X}) = \Gamma(X, \mathcal{Q}_{\mathbb{X}})$. Note that $Q(\mathbb{X})$ is a semisimple $Q(X)$-algebra and for every closed point $x \in X$ the ring $\mathcal{O}_{\mathbb{X},x}$ is an $\mathcal{O}_{X,x}$-order in this algebra. Since X is excellent, the set $\mathrm{Sing}(\mathbb{X})$ of such closed points $x \in X$ that this order is not maximal is finite (it follows from [4, Ch. 6]).

Theorem 7. *Let $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$ and $\mathbb{Y} = (X, \mathcal{O}_{\mathbb{Y}}))$ be two central non-commutative curves with the same central curve X. They are centrally Morita equivalent if and only if the following conditions are satisfied:*

- *the semisimple $Q(X)$-algebras $Q(\mathbb{X})$ and $Q(\mathbb{Y})$ are centrally Morita equivalent;*
- $\mathrm{Sing}(\mathbb{X}) = \mathrm{Sing}(\mathbb{Y})$*;*
- *for every $x \in \mathrm{Sing}(\mathbb{X})$ the $\mathcal{O}_{X,x}$-orders $\mathcal{O}_{\mathbb{X},x}$ and $\mathcal{O}_{\mathbb{Y},x}$ (or, which is the same, their $\mathfrak{m}_x$-completions) are centrally Morita equivalent.*

References

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