On ternary assymetric medial top-quasigroups

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Let Q be an m element set. A ternary operation f defined on Q is called *invertible* and the pair $(Q; f)$ is a *quasigroup* of the order m, if for every a, b of Q the terms $f(x, a, b)$, $f(a, x, b)$, $f(a, b, x)$ define permutations of Q. To each ternary quasigroup $(Q; f)$ of the order m there corresponds a Latin cube of order m, i.e., a 3-dimensional array on m distinct symbols from Q, each of which occurs exactly once in any line of the array.

A triplet (f_1, f_2, f_3) of ternary operations is called *orthogonal* [1], if for all $a_1, a_2, a_3 \in Q$ the system

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases}$$

has a unique solution, i.e., superimposition of the corresponding cubes gives a cube such that every triplet of elements of Q appears exactly once in it.

Geometric interpretation of orthogonality is its relationships with geometric nets. This application is well-studied for binary operations and the respective k-nets, projective and affine planes (see for example [2], [3]). Relationships between t-tuples of orthogonal n-ary quasigroups of order m and (t, m, n)-nets were studied in [4], [5], [6]. The respective nets have the same combinatorial and algebraic properties.

For every permutation $\sigma \in S_4$ a *σ-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation f is defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4.$$

In particular, a σ-parastrophe is called:

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- *principal* if $4\sigma = 4$.

Therefore, each ternary operation has at most $4! = 24$ parastrophes; among them $3! = 6$ principal parastrophes. An invertible operation and the respective quasigroup are called *assymetric* if all its parastrophes are different. A quasigroup is called *totally parastrophic orthogonal* (*top-quasigroup*), if each triplet of its different parastrophes are orthogonal. Binary assymetric top-quasigroups were studied in [7], for ternary case the following statements are true.

Theorem 1 ([8]). *A quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$ such that*

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a, \tag{1}$$

where φ_1, φ_2, φ_3 are pairwise commuting automorphisms of $(Q; +)$ and $a \in Q$.

Theorem 2. *Let $(Q; f)$ be a medial ternary quasigroup $(Q; f)$ with (1) and τ_1, τ_2, $\tau_3 \in S_4$. The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are orthogonal if and only if the determinant*

$$\begin{vmatrix} \varphi_{1\tau_1} & \varphi_{2\tau_1} & \varphi_{3\tau_1} \\ \varphi_{1\tau_2} & \varphi_{2\tau_2} & \varphi_{3\tau_2} \\ \varphi_{1\tau_3} & \varphi_{2\tau_3} & \varphi_{3\tau_3} \end{vmatrix}$$

is an automorphism of the group $(Q; +)$, where $\varphi_4 := J$ and $J(x) := -x$.

Note, that the pairwise commuting automorphisms φ_1, φ_2, φ_3, J generate a commutative subring K of the ring $End(Q; +)$. Let $\vec{\nu} := (\nu_1, \nu_2, \nu_3)$ be a triplet of injections of the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4\}$. The polynomial

$$d_{\vec{\nu}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{vmatrix} \gamma_{1\nu_1} & \gamma_{2\nu_1} & \gamma_{3\nu_1} \\ \gamma_{1\nu_2} & \gamma_{2\nu_2} & \gamma_{3\nu_2} \\ \gamma_{1\nu_3} & \gamma_{2\nu_3} & \gamma_{3\nu_3} \end{vmatrix}$$

over the commutative ring K will be called *invertible-valued* over a set $H \subseteq K$, if all its values are automorphisms of the group $(Q; +)$ when the variables γ_1, γ_2, γ_3, γ_4 take their values in H.

Theorem 3. *A ternary medial quasigroup $(Q; f)$ with (1) is a top-quasigroup if and only if each polynomial $d_{\vec{\nu}}$ is invertible-valued over the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_4 := J$.*

Theorem 4 ([9])**.** *A ternary medial assymetric top-quasigroup over a cyclic group of the order m exists if and only if the least prime factor of m is greater than 19.*

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