The geometrical properties of degenerations of curves and surfaces

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In this talk, we will mainly discuss the topology and arithmetic properties of degenerations of curves and surfaces. First, we investigate the influences of the base points of cubic pencils on the Mordell-Weil groups in this part. We pay attention to 8, 7 ,6 and 5 base points in general position for such cubic pencil, and classify these cubic pencils. And we give the following theorem:

Theorem 1. *This is the main theorem (taken from [2]).*
Given $n\ (=8,7,6,5)$ *points in general position in* $\mathbb{P}^2$, $S: sH_1+tH_2=0,\quad [s,t]\in\mathbb{P}^1$ *is a cubic pencil with* $n\ (=8,7,6,5)$ *simple base points. Then, the Mordell-Weil groups of the fibrations are isomorphic to two types respectively:*

$$E_8:\quad y^2=x^3+x(\sum_{i=0}^{3}p_it^i)+\sum_{i=0}^{3}q_it^i+t^5,\quad y^2=x^3+t^2x^2+x(\sum_{i=0}^{2}p_it^i)+\sum_{i=0}^{4}q_it^i+t^5 \tag{1}$$

$$E_7^\vee:\quad y^2=x^3+x(p_0+p_1t+t^3)+\sum_{i=0}^{4}q_it^i,\quad y^2+txy=x^3+x(\sum_{i=0}^{2}p_it^i)+\sum_{i=0}^{3}q_it^i-t^4 \tag{2}$$

$$E_6^\vee:\quad y^2+t^2y=x^3+x(\sum_{i=0}^{2}p_it^i)+(\sum_{i=0}^{2}q_it^i),\quad y^2+txy=x^3+x(\sum_{i=0}^{2}p_it^i)+(\sum_{i=0}^{3}q_it^i) \tag{3}$$

$$D_5^\vee:\quad y^2+p_5xy=x^3+p_4tx^2+(p_8t^2+p_2t^3)x+p_6t^4+t^5 \tag{4}$$

A Del Pezzo surface X is either $\mathbb{P}^1\times\mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ in $m\ (m=1,\cdots,8)$ points in general position. The degree d of X is defined to be $d=9-m$. As an application, we give a new proof of the number of (-1) curves in Del Pezzo surfaces.

Theorem 2. *The number of* (-1) *curves in Del Pezzo surfaces of degree* $1,2,3,4$ *is 240, 56, 27 and 16 respectively.*

In the second part, we talk about the surfaces of minimal degree in $\mathbb{P}^n$. In fact, the degree of such surface is $n-1$. The fundamental group of Galois cover of surface is an important invariant of the moduli space of such surfaces. In [1], we use the tools of degenerations of surfaces to prove the following theorem:

Theorem 3. *The Galois cover of the surface of minimal degree is simple-connected and general type.*

In the end, we give an open question:

Question: It is well known that the fundamental groups of most surfaces of general type are non commutative. But it is not easy to find concrete examples of such surfaces. Let a_k be a series of integral number whose limit is infinity. How to give a series of surfaces of degree a_k whose the fundamental groups of Galois covers are all non commutative?

References

[1] Amram, M., Gong, C., Mo, J.-L., *On the Galois covers of degenerations of surfaces of minimal degree*, appear in Mathematische Nachrichten, 2021.
[2] Mo, J.-L., *The Mordell-Weil groups of cubic pencils*, reprint.