About solvability of the matrix equation $AX = B$ over Bezout domains

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Let R be a Bezout domain with identity $e \neq 0$, i.e. R is an integral domain in which every finite generated ideal is principal. Further, let $\mathrm{R}_{m,n}$ denote the set of $m \times n$ matrices over R, and $GL(n, \mathrm{R})$ be the set of $n \times n$ invertible matrices over R. In what follows, I_n is the identity $n \times n$ matrix, $0_{m,k}$ is the zero $m \times k$ matrix, $d_i(A) \in \mathrm{R}$ is an ideal generated by the $i-$th order minors of the matrix $A \in \mathrm{R}_{m,n}$, $i = 1, 2, \dots, \min\{m, n\}$.

Let $A \in \mathrm{R}_{m,n}$ and $B \in \mathrm{R}_{m,k}$ be nonzero matrices. Consider the nonhomogeneous matrix equation

$$AX = B, \tag{1}$$

where X is unknown matrix in $\mathrm{R}_{n,k}$. Denote by $A_B = \begin{bmatrix} A & B \end{bmatrix} \in \mathrm{R}_{m,n+k}$ the extended matrix of the linear equations (1). It is known (see [1], [3], [4], [6]) that the equation (1) over a Bezout domain R is solvable if and only if $\operatorname{rank} A = \operatorname{rank} A_B = r$ and $d_i(A) = d_i(A_B)$ for all $i = 1, 2, \dots, r$.

The problem of solvability of the equation (1) has drawn the attention of many mathematicians (see [1]–[12] and references therein). This is explained not only by the theoretical interest to this problem ([1], [3], [4], [6], [8]–[11]), but also by the existence of numerous applied problems connected with the necessity of solution of linear matrix equations ([2], [5], [7], [12]). It may be noted, that the equation (1) over Bezout domains is important in automatic control theory [2].

1. On application of the Hermite Normal Form. In the Bezout domain R we fix a set of non-associated elements $\widetilde{\mathrm{R}}$. Every non-associated element $a \in \widetilde{\mathrm{R}}$ we associated with a complete system of residues modulo ideal (a). Let $A \in \mathrm{R}_{m,n}$ and $\operatorname{rank} A = r$. Further, we assume that the first row of the matrix A is nonzero. For the matrix A there exists $W \in GL(n, \mathrm{R})$ such that

$$AW = H_A = \begin{bmatrix} H_1 & 0_{m_1,n-1} \\ H_2 & 0_{m_2,n-2} \\ \dots & \dots \\ H_r & 0_{m_r,n-r} \end{bmatrix} = \begin{bmatrix} H(A) & 0_{m,n-r} \end{bmatrix}$$

is a lower block-triangular matrix, where $H(A) \in \mathrm{R}_{m,r}$, $H_1 = \begin{bmatrix} h_1 \\ * \end{bmatrix} \in \mathrm{R}_{m_1,1}$, $H_2 = \begin{bmatrix} h_{21} & h_2 \\ * & * \end{bmatrix} \in \mathrm{R}_{m_2,2}$, $\dots$, $H_r = \begin{bmatrix} h_{r1} & \dots & h_{r,r-1} & h_r \\ * & * & * & * \end{bmatrix} \in \mathrm{R}_{r,r}$ and $m_1 + m_2 + \dots + m_r = m$. The elements h_i belong to the set of non-associated elements $\widetilde{\mathrm{R}}$ for all $i = 1, 2, \dots, r$. Moreover, in the first rows $\begin{bmatrix} h_{i1} & \dots & h_{i,i-1} & h_i \end{bmatrix}$ of the matrices H_i, $i \geq 2$, the elements h_{ij} belong to a complete system of residues modulo ideal (h_i) for all $j = 1, 2, \dots, i-1$. The lower block-triangular matrix H_A is called the (right) Hermite normal form of the matrix A and it is uniquely defined for A (see [3]).

In this parch we propose necessary and sufficient conditions of solvability for the equation (1) over a Bezout domain in terms of the Hermite normal forms of $m \times (n+k)$ matrices $\begin{bmatrix} A & 0_{m,k} \end{bmatrix}$ and $\begin{bmatrix} A & B \end{bmatrix}$. A method for finding its solutions is also given. In what follows, we assume that the fest row of the matrix A is nonzero.

Theorem 1. *Let $A \in \mathrm{R}_{m,n}$ and $B \in \mathrm{R}_{m,k}$. The matrix equation $AX = B$ is solvable over a Bezout domain* R *if and only if the Hermite normal forms of matrices $\begin{bmatrix} A & 0_{m,k} \end{bmatrix}$ and $\begin{bmatrix} A & B \end{bmatrix}$ are coincide.*

It is easy to see that matrix equation (1) is solvable over a Bezout domain R if and only if the matrix equation $H(A)Y = B$ is solvable over R. Let $Y_0 \in \mathrm{R}_{r,k}$ be the solution of $H(A)Y = B$. Then for arbitrary matrix $P \in \mathrm{R}_{n-r,k}$ the matrix $X_P = W^{-1}\begin{bmatrix} Y_0 \\ P \end{bmatrix}$ is a general solution of equation (1). Theoretically speaking, the solution $X_0 = W^{-1}\begin{bmatrix} Y_0 \\ 0_{m-r,n} \end{bmatrix}$ of equation (1) can be written as the matrix expression $X_0 = TX_P$, where $T \in \mathrm{R}_{n,n}$. Thus, X_P is the right divisor of X_0 for an arbitrary matrix $P \in \mathrm{R}_{n-r,k}$. Given the solution X_0, we determine all possible ranks of other solutions of the equation (1), i.e. $\mathrm{rank}B \le \mathrm{rank}X_P \le n + \mathrm{rank}B - \mathrm{rank}A$.

2. A method of matrix transformations. In this part we apply matrix transformations for established conditions under which the equation (1) is solvable.

Let $A \in \mathrm{R}_{m,n}$ and $B \in \mathrm{R}_{m,k}$ be nonzero matrices and let $\mathrm{rank}\, A = r \ge 1$. For A there exist matrices $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$. It is clear that $\det C = c \ne 0$. In what follows $C^* = \mathrm{Adj}\, C$ means the classical adjoint matrix of the matrix C, i.e. $C^*C = cI_r$. Based on the above, we obtain the following theorem.

Theorem 2. *The matrix equation $AX = B$ is solvable over a Bezout domain* R *if and only if* $UB = \begin{bmatrix} D \\ 0_{m-r,k} \end{bmatrix}$, *where $D \in \mathrm{R}_{r,k}$, and $C^*D = cG$, where $G \in \mathrm{R}_{r,k}$.*

If the equation $AX = B$ is solvable, then for arbitrary matrix $Q \in \mathrm{R}_{m-r,k}$ the matrix $X_Q = U^{-1}\begin{bmatrix} G \\ Q \end{bmatrix}$ is a general solution of equation $AX = B$.

From Theorem 2 we obtain the following comment. Let $A, B \in \mathrm{R}_{m,n}$ be nonzero matrices and let $\mathrm{rank}\, A < n$. Suppose the matrix equation $AX = B$ is solvable and $X_Q \in \mathrm{R}_{n,n}$ is its general solution. Then $AX = B$ has solutions $\widetilde{X}_i \in \mathrm{R}_{n,n}$, $i = 1, 2, \ldots$, such that $X_Q = \widetilde{X}_i T_i$, where $T_i \in \mathrm{R}_{n,n}$.

Presented results above can be extended to linear nonhomogeneous equations over commutative rings of a more general algebraic nature.

References

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