Hopf-Rinow theorem of sub-Finslerian geometry

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The sub-Finslerian geometry means that the metric F is defined only on a given subbundle of the tangent bundle, called a horizontal bundle. In the paper, a version of the Hopf-Rinow theorem is proved in the case of sub-Finslerian manifolds, which relates the properties of completeness, geodesically completeness, and compactness. The sub-Finsler bundle, the exponential map and the Legendre transformation are deeply involved in this investigation.

We construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. We introduce the notion of an exponential map in sub-Finsler geometry. At the end, our main theorem is stated and proved.

Theorem 1. *Let $(M, \mathcal{D}, F)$ be any connected sub-Finsler manifold, where $\mathcal{D}$ is bracket generating distribution. The following conditions are equivalent:*

(i) *The metric space (M, d) is forward complete.*
(ii) *The sub-Finsler manifold $(M, \mathcal{D}, F)$ is forward geodesically complete.*
(iii) *$\Omega_x^* = \mathcal{D}_x^*$, additionally, the exponential map is onto if there are no strictly abnormal minimizers.*
(iv) *Every closed and forward bounded subset of (M, d) is compact.*

Furthermore, for any $x, y \in M$ there exists a minimizing geodesic γ joining x to y, i.e. the length of this geodesic is equal to the distance between these points.

References

[1] A. Agrachev, D. Barilari, U. Boscain. *A Comprehensive Introduction to Sub-Riemannian geometry*, Cambridge Studies in Advanced Mathematics, 2019.
[2] L. M. Alabdulsada, L. Kozma. *On the connection of sub-Finslerian geometry*, Int. J. Geom. Methods Mod. Phys. **16**, No. supp02, 1941006 (2019)
[3] L. M. Alabdulsada. *A note on the distributions in quantum mechanical systems*, J. Phys.: Conf. Ser. **1999**, (2021), 012112
[4] L. M. Alabdulsada. *Sub-Finsler geometry and nonholonomic mechanics*, submitted.
[5] D. Bao, S.-S. Chern, Z. Shen. *An Introduction to Riemann-Finsler geometry*, Graduate Texts in Mathematics 200, Springer-Verlag, New York, (2000).
[6] O. Calin, D. Chang. *Subriemannian geometry, a variational approach*, J. Differential Geom. **80** (2008), no. 1, 23–43.
[7] P. do Carmo. *Riemannian geometry. Mathematics: Theory & Applications*, Birkhäuser Boston, Inc., Boston, MA, (1992).
[8] W.-L. Chow. *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117** (1939) 98-105.
[9] R. Montgomery. *A Tour of Subriemannian Geometries, their Geodesics and Applications*, Mathematical Surveys and Monographs 91. Amer. Math. Soc. Providence, RI, (2002).
[10] B. O'Neill. *Semi-Riemannian Geometry, With applications to Relativity*, Pure and Applied Mathematics, 103. Academic Press, Inc. New York, (1983).
[11] C. B. Rayner. *The exponential map for the Lagrange problem on differentiable manifolds*, Philosophical Transactions of the Royal Society of London, Series A, Mathematical and Physical Sciences Vol. 262, No. 1127 (1967), pp. 299-344.
[12] L. Rifford. *Sub-Riemannian geometry and optimal transport*, Springer, (2014).
[13] R. Strichartz. *Sub-Riemannian geometry.* J. Differ. Geom. **24** (1986), 221-263; correction ibid. **30** (1989), 595-596.