Geometric properties of interception curves

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In this study, a plane curve, which was named as *Interception Curve*, was discussed. This curve can be defined in the following way. Suppose one point moves with constant velocity along a straight line, and another point, at the beginning one unit apart from the line and the first point on this line, moves with the same constant speed so that it always stays on a line passing through the first point and the initial position of the second point. This plane curve appears in problems related to the interception of high-speed targets by beam rider missiles (hence the name *Interception Curve*) [2, 5]. This curve was also mentioned in [4, 6, 1]. In [3], at Sect. 1.460 and Sect. 1.507), some methods based on polar and Cartesian coordinates were proposed to find an explicit representation for this curve.

Problem 1. If two points $P(x, y)$ and Q, initially at $O(0, 0)$ and $A(1, 0)$, respectively, move uniformly so that Q is on the line $x = 1$, and P is on the ray OQ then what curve does the point P draw?

Answer. Let us use polar coordinates $r = |OP|$ and $\angle AOQ = \theta$. We obtain ordniary differential equation

$$r(\theta)^2 + (r'(\theta))^2 = \frac{1}{\cos^4\theta}, \tag{1}$$

with initial condition $r(0) = 0$. Note that in the cartesian coordinates, (1) can be written as

$$x^2\sqrt{1 + (y'(x))^2} = y'x - y, \tag{2}$$

with initial condition $y(0) = 0$. By solving this equation, we obtain the parametrization (cf. [3], Sect. 1.507, where the roles of x and y are interchanged)

$$\begin{cases} x(p) = \frac{1}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}, \\ y(p) = \frac{\sqrt{p^2-1}}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}} - \left(\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}\right)^2 \ (p \geq 1). \end{cases} \tag{3}$$

Using all these, the following results are obtained:

Theorem 2. *Suppose that U is the y intercept of the tangent line of the curve (3) at the point P, and this tangent line intersects the line $x = 1$ at point and T. Then*

(1) $x \cdot |UP| = |OU|$,
(2) $\sin\angle QPT = \frac{x^2}{|OP|} = \frac{x}{|OQ|}$,

where x is the abscissa of the point $P(x, y)$.

Theorem 3. *Consider intersection point M of mid-perpendicular of OP and the line perpendicular to UT at the point P. Similarly, conside intersection point N of mid-perpendicular of OQ and the line perpendicular to QT at the point Q. Then the points M and N are equidistant from the point O i.e. $MO{=}NO$.*

The following result shows that there is a connection between the interception curve and Gauss's constant G defined by the arithmetic–geometric mean.

Theorem 4.

$$\lim_{x\to 1^-} |PQ| = \frac{1}{4G^2}.$$

Problem 5. Suppose that two points P and Q, at the beginning at $B(0,0,1)$ and $A(1,0,0)$, respectively, move uniformly so that Q is on the equator $z=0$, $x^2+y^2=1$ of sphere $x^2+y^2+z^2=1$ with center $O(0,0,0)$, and P is on the meridian through B and Q of the sphere. What curve does the point P draw?

Answer. We can use spherical coordinates to describe this curve: $\angle AOQ=\theta$ and $\angle POB=\phi$. Since $\rho=|OP|=1$, for the coordinates of point $P(x,y,z)$, we can write $x=\cos\theta\sin\phi$, $y=\sin\theta\sin\phi$, and $z=\cos\phi$, where we think of $\phi=\phi(\theta)$ as a function of θ. For this curve we obtain

$$\phi=\tan^{-1}\sinh\theta. \tag{4}$$

Note that (4), which can also be expressed as $\sin\phi=\tanh\theta$, is sometimes called Gudermannian function $\mathrm{gd}(x)$. For the curve defined by (4) the following results are obtained.

Theorem 6. $\lim_{\theta\to\infty}|PQ|=0$.

In the following, we will use notation $\widehat{XY}$ for the spherical distance between points X and Y on a sphere.Of course, for a unit sphere with center O, $\widehat{XY}=\angle XOY$.

Theorem 7. *If a great circle is tangent to the curve (4) at point P, intersects the equator at point T, then*

(1) $\widehat{PT}=\frac{\pi}{2}-\widehat{TQ}$,
(2) $\widehat{TQ}<\widehat{PT}$, *and* $\lim_{\theta\to\infty}\widehat{TQ}=\lim_{\theta\to\infty}\widehat{PT}=\frac{\pi}{4}$.
(3) $\angle BPT=\pi-\widehat{BP}$.

Theorem 8. *If a small spherical circle through point B is tangent to the curve (4) at point P, then its spherical radius R satisfies* $\tan R=\frac{1}{2}\sec^2\frac{1}{2}\widehat{BP}$.

We can prove some of these results also using simpler plane and spherical geometry methods, which are interesting on their own. It can be shown that the results agree with the angle-preserving property of Mercator and Stereographic projections. The Mercator and Stereographic projections also reveal the symmetry of this curve with respect to Spherical and Logarithmic Spirals.

References

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