Surfaces with zero mean curvature vector in 4-dimensional spaces

Naoya Ando
(Faculty of Advanced Science and Technology, Kumamoto University, 2–39–1 Kurokami, Kumamoto 860–8555 Japan)
E-mail: andonaoya@kumamoto-u.ac.jp

Let (N, h) be an oriented Riemannian 4-manifold. Let $\bigwedge^2 TN$ be the 2-fold exterior power of the tangent bundle TN of N. Then $\bigwedge^2 TN$ is a vector bundle of rank 6 over N and Hodge's $*$-operator gives a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$ by two subbundles $\bigwedge^2_\pm TN$ of rank 3. The twistor spaces associated with N are the sphere bundles in $\bigwedge^2_\pm TN$ and denoted by $U\left(\bigwedge^2_\pm TN\right)$. We can refer to [5] for twistor spaces. Let M be a Riemann surface and $F : M \longrightarrow N$ a conformal and minimal immersion. Let F^*TN be the pull-back bundle on M by F. Then F gives its twistor lifts, which are sections of $U\left(\bigwedge^2_\pm F^*TN\right)$. Let σ be the second fundamental form of F. Let w be a local complex coordinate of M and set $\sigma_{ww} := \sigma(\partial/\partial w, \partial/\partial w)$. Then $h(\sigma_{ww}, \sigma_{ww})dw^4$ does not depend on the choice of w and therefore F gives a complex quartic differential Q on M. If N is a 4-dimensional space form, then Q is holomorphic. Isotropicity of F is given by $Q \equiv 0$ and this condition is equivalent to horizontality of a twistor lift of F ([6], [4]).

Let (N, h) be an oriented neutral 4-manifold. Then the metric $\hat{h}$ of $\bigwedge^2 TN$ induced by h has signature $(2, 4)$. We have a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$, and the restriction of $\hat{h}$ on each of $\bigwedge^2_\pm TN$ has signature $(1, 2)$. The space-like (or hyperbolic) twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ such that fibers are hyperboloids of two sheets, and denoted by $U_+\left(\bigwedge^2_\pm TN\right)$. We can refer to [3] for space-like twistor spaces. Let M be a Riemann surface and $F : M \longrightarrow N$ a space-like and conformal immersion with zero mean curvature vector. Then F gives its space-like twistor lifts, which are sections of $U_+\left(\bigwedge^2_\pm F^*TN\right)$. Let Q be a complex quartic differential on M defined by F as in the previous paragraph. Then isotropicity of F is given by $Q \equiv 0$, which is equivalent to horizontality of a space-like twistor lift of F ([1]).

Let (N, h) be as in the previous paragraph. The time-like twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ such that fibers are hyperboloids of one sheet, and denoted by $U_-\left(\bigwedge^2_\pm TN\right)$. We can refer to [7], [8] for time-like twistor spaces. Let M be a Lorentz surface, which is an analogue of a Riemann surface and a two-dimensional manifold equipped with a holomorphic system of paracomplex coordinate neighborhoods. Let $F : M \longrightarrow N$ be a time-like and conformal immersion with zero mean curvature vector. Then F gives its time-like twistor lifts $\Theta_{F,\pm}$, which are sections of $U_-\left(\bigwedge^2_\pm F^*TN\right)$. Let Q be a paracomplex quartic differential on M defined by F. Then isotropicity of F is given by $Q \equiv 0$. If one of $\Theta_{F,\pm}$ is horizontal, then $Q \equiv 0$ ([1]), while $Q \equiv 0$ does not necessarily mean the horizontality of $\Theta_{F,\pm}$: it is possible that although F is isotropic, the covariant derivatives of $\Theta_{F,\pm}$ are not zero but light-like. The covariant derivatives of $\Theta_{F,\pm}$ are light-like or zero if and only if one of the following holds: (a) the shape operator of a light-like normal vector field vanishes and then Q vanishes; (b) the shape operator of any normal vector field is light-like or zero, and then Q is null or zero ([2]). The conformal Gauss maps of time-like surfaces of Willmore type in 3-dimensional Lorentzian space forms with zero holomorphic quartic differential satisfy Condition (a) ([1]). If N is a 4-dimensional neutral space form, then we can characterize surfaces with Condition (b), based on the Gauss-Codazzi-Ricci equations ([2]).

References

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