Edge resolvability and topological characteristics of zero-divisor graphs

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Definition 1. (Zero- Divisor Graph) Zero-divisor graph is a geometric representation of a commutative ring. Zero-divisor graph of ring R is denoted by $\Gamma(R) = (V(\Gamma), E(\Gamma))$, defined by a graph whose vertices are all elements of the zero-divisor set of a ring R, and two distinct vertices z_1 and z_2 are adjacent if and only if $z_1.z_2 = 0$.

Definition 2. (Metric Dimension) Let $G = (V(G), E(G))$ be a graph, and $S \subset V(G)$ be an ordered subset of the principal nodes set, defined as $s = \{\aleph_1, \aleph_2, \aleph_3, ..., \aleph_k\}$. Let $\aleph$ be any principal node in $V(G)$. The identification of a principal node $\aleph$ with respect to S is a k-ordered distance set $\big(d(\aleph, \aleph_1), d(\aleph, \aleph_2), ..., d(\aleph, \aleph_k)\big)$. If each principal node for $V(G)$ has a unique identification according to ordered subset S, then this subset is called resolving set of graph G. The minimum number of elements in the subset S is called the metric dimension of G.

Definition 3. [1] (Edge Metric Dimension) If in a simple and connected graph G, the distinct edges of G have distinct representation with respect to an ordered subset R of vertices of G, then S is known as edge resolving set of G. The minimal edge resolving set of G is called edge metric basis, and its cardinality is called edge metric dimension of G. The edge metric dimension of graph G is denoted by $edim(G)$.

These are some important findings

Theorem 4. [2] *For a graph G, we have*

$$edim(G) = \begin{cases} 1, & \textit{iff } G = P_n, \ \textit{(Path graph)} \\ n-1, & \textit{iff } G = K_n, \ \textit{(Complete graph)} \\ 2, & \textit{if } G = C_n, \ \textit{(Cycle graph)} \\ n-2, & \textit{if } G \cong K_{1,n} \ \textit{(except } K_{1,1}), \ \textit{or a bipartite graph} \end{cases}$$

Theorem 5. [3] *The diameter of $\Gamma(R) \leq 3$, where R is a commutative ring.*

Theorem 6. *The edge metric dimension of the zero-divisor graph of R is finite iff R is finite, where $R - \{0\}$ is a commutative ring but not an integral domain.*

Theorem 7. *For the ring $\mathbb{Z}_m$, where $m \geq 1$, we have*

$$edim\left(\Gamma(\mathbb{Z}_m)\right) = \begin{cases} undefined, & \textit{if } m = p \textit{ is a prime} \\ p-2, & \textit{if } m = p^2 \textit{and } p > 2 \end{cases}$$

Theorem 8. *Consider the ring $\mathbb{Z}_m$, where $m \geq 1$, we have*

$$edim\left(\Gamma(\mathbb{Z}_m)\right) = \begin{cases} undefined, & \textit{if } m = 2p, \textit{where } p \textit{ is an even prime} \\ p-2, & \textit{if } m = 2p, \textit{where } p \textit{ is an odd prime} \end{cases}$$

Theorem 9. *Consider the ring $\mathbb{Z}_m[i]$, where p is a prime then*

$$edim\left(\Gamma(\mathbb{Z}_m[i])\right) = \begin{cases} undefined, & \text{if } m = 2 \\ p^2 - 2, & \text{if } m = p^2 \end{cases}$$

Theorem 10. *Consider the ring $\mathbb{Z}_p[i]$, where p is a prime. If $p \equiv m(mod\, 4)$ then*

$$edim\left(\Gamma(\mathbb{Z}_p[i])\right) = \begin{cases} 2p - 4, & \text{if } m = 1 \\ undefined, & \text{if } m = 2 \\ undefined, & \text{if } m = 3 \end{cases}$$

Theorem 11. *Consider the ring $\mathbb{Z}_m[i]$, then Zagreb first index (M_1)*

$$M_1\left(\Gamma(\mathbb{Z}_m[i])\right) = \begin{cases} 1, & \text{if } m = 2 \\ (p^2 - 2)^2(p^2 - 1), & \text{if } m = p^2 \text{where } p \text{ is a prime} \end{cases}$$

Theorem 12. *Consider the ring $\mathbb{Z}_p[i]$, where p is a prime. If $p \equiv m(mod\, 4)$ then*

$$M_1\left(\Gamma(\mathbb{Z}_p[i])\right) = \begin{cases} (2p - 2)(p - 1)^2, & \text{if } m = 1 \\ undefined, & \text{if } m = 2 \\ 0, & \text{if } m = 3 \end{cases}$$

Remark 13. This article examines the edge metric dimension and topological nature of $\Gamma(R)$. We have looked closely at edge metric dimension of integers modulo m, and Gaussian integers modulo m. We also discovered the first Zagreb index, second Zagreb index, and Sombor index of the zero divisor graph of the Gaussian integers modulo m. These findings are helpful for researching the structural characteristics of rings and chemical compounds.

References

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