From minimality to maximality via metric reflection

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In 1934 Đuro Kurepa [4] introduced the pseudometric spaces which, unlike metric spaces, allow the zero distance between different points.

Definition 1. Let X be a set and let $d\colon X^2 \to \mathbb{R}$ be a non-negative, symmetric function such that $d(x,x)=0$ for every $x \in X$. The function d is a *pseudometric* on X if it satisfies the triangle inequality.

If d is a pseudometric on X, we say that (X,d) is a *pseudometric space*.

Definition 2 ([3])**.** Let (X,d) and (Y,ρ) be pseudometric spaces. The spaces (X,d) and (Y,ρ) are *combinatorially similar* if there exist bijections $\Psi\colon Y \to X$ and $f\colon d(X^2) \to \rho(Y^2)$ such that $\rho(x,y) = f\big(d(\Psi(x),\Psi(y))\big)$ for all x, $y \in Y$. In this case, we will say that $\Psi\colon Y \to X$ is a *combinatorial similarity* and that (X,d) and (Y,ρ) are combinatorially similar pseudometric spaces.

Definition 3. Let (X,d) be a pseudometric space. A bijection $f : X \to X$ is a *pseudoidentity* if the equality $d(x,f(x))=0$ holds for every $x \in X$.

The groups of all combinatorial self-similarities and all pseudoidentities of a pseudometric space (X,d) will be denoted by $\mathbf{Cs}(X,d)$ and $\mathbf{PI}(X,d)$ respectively. Thus, for every pseudometric space (X,d) we have $\mathbf{PI}(X,d) \subseteq \mathbf{Cs}(X,d) \subseteq \mathbf{Sym}(X)$, where $\mathbf{Sym}(X)$ is the symmetric group of all permutations of the set X.

For every nonempty pseudometric space (X,d), we define a binary relation $\overset{0(d)}{=}$ on X by

$$(x \overset{0(d)}{=} y) \Leftrightarrow (d(x,y)=0), \quad \text{for all } x, y \in X.$$

Proposition 4. *Let X be a nonempty set and let $d\colon X^2 \to \mathbb{R}$ be a pseudometric on X. Then $\overset{0(d)}{=}$ is an equivalence relation on X and, in addition, the function δ_d,*

$$\delta_d(\alpha,\beta) := d(x,y), \quad x \in \alpha \in X/\overset{0(d)}{=}, \quad y \in \beta \in X/\overset{0(d)}{=},$$

is a correctly defined metric on the quotient set $X/\overset{0(d)}{=}$.

In what follows we will say that the metric space $(X/\overset{0(d)}{=}, \delta_d)$ is the *metric reflection* of (X,d).
Let us define a class $\mathcal{IP}$ of pseudometric spaces as follows.

Definition 5. A pseudometric space (X,d) belongs $\mathcal{IP}$ if the equalities

$$\mathbf{Cs}(X,d) = \mathbf{PI}(X,d) \quad \text{and} \quad \mathbf{Cs}(X/\overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/\overset{0(d)}{=}) \quad \text{hold.}$$

Our main goal is to describe the structure of pseudometric spaces belonging to $\mathcal{IP}$. To do this, we introduce into consideration pseudometric generalizations of some well-known classes of metric spaces.
Let (X,d) be a metric space. Recall that the metric d is said to be *strongly rigid* if, for all x, y, u, $v \in X$, the condition $d(x,y) = d(u,v) \neq 0$ implies ($x=u$ and $y=v$) or ($x=v$ and $y=u$). The

discrete metric d on X is defined by $d(x,y) = k$, if $x \neq y$ and $d(x,y) = 0$, if $x = y$ for any $x, y \in X$ and arbitrary fixed $k > 0$.

Definition 6. Let (X, d) be a pseudometric space. Then d is *discrete* (*strongly rigid*) if all metric subspaces of (X, d) are discrete (strongly rigid).

Definition 7. A pseudometric space (X, d) is a *pseudorectangle* if all three-point metric subspaces of (X, d) are strongly rigid and isometric and, in addition, there is a four-point metric subspace Y of (X, d) such that for every $x \in X$ we can find $y \in Y$ satisfying $d(x, y) = 0$.

Let X be a nonempty set and $P = \{X_j : j \in J\}$ be a set of nonempty subsets of X. The set P is a *partition* of X with the *blocks* X_j, $j \in J$, if $\cup_{j\in J} X_j = X$ and $X_{j_1} \cap X_{j_2} = \varnothing$ for all distinct $j_1, j_2 \in J$.

Now we are ready to characterize the pseudometric spaces satisfying equality

$$\mathbf{Cs}(X/ \overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/ \overset{0(d)}{=}) \quad \text{(see [1] for more details).}$$

The following theorem is, in fact, a pseudometric modification of the main result of [2].

Theorem 8. *Let (X, d) be a nonempty pseudometric space. Then the following statements are equivalent:*

(i) *At least one of the following conditions has been fulfilled:*
- (i_1) (X, d) *is strongly rigid;*
- (i_2) (X, d) *is discrete;*
- (i_3) (X, d) *is a pseudorectangle.*

(ii) *The equality*

$$\mathbf{Cs}(X/ \overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/ \overset{0(d)}{=})$$

holds.

The next theorem can be considered as one of the main results of our work.

Theorem 9. *Let (X, d) be a nonempty pseudometric space and let $\{X_j : j \in J\}$ be a partition of X corresponding the equivalence relation $\overset{0(d)}{=}$. Then $(X, d) \in \mathcal{IP}$ if and only if*

$$|X_{j_1}| \neq |X_{j_2}|$$

holds whenever $j_1, j_2 \in J$ are distinct and, in addition, at least one of the following conditions has been fulfilled:

(i) (X, d) *is strongly rigid;*
(ii) (X, d) *is discrete;*
(iii) (X, d) *is a pseudorectangle.*

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