Thurston norm and Euler classes of bounded mean curvature foliations on hyperbolic 3-Manifolds

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Let M be a closed, oriented 3-manifold, and suppose that M contains no non-separating 2 - spheres or tori. For example, M is a closed oriented hyperbolic 3-Manifold.

The Thurston norm on $H_2(M,\mathbb{Z})$ is defined as follows ([1]):

$$||a||_{Th} = inf\{\chi_-(\Sigma)|\ \Sigma \text{ is an embedded oriented surface representing } a \in H_2(M,\mathbb{Z})\}, \qquad (1)$$

where $\chi_-(\Sigma) = max\{-\chi(\Sigma), 0\}$. Recall that $\chi(\Sigma) = 2 - 2g$ denotes the Euler characteristic of a surface Σ of genus g. When Σ is not connected, define $\chi_-(\Sigma)$ to be the sum $\chi_-(\Sigma_1) + \cdots + \chi_-(\Sigma_k)$, where $\Sigma_i,\ i = 1, \ldots, k$ are the connected components of Σ. As Thurston showed, the Thurston norm can be extended in a unique way to the norm in $H_2(M,\mathbb{R})$.

The dual Thurston norm can be defined on $H^2(M,\mathbb{R})$ by the formula

$$||\alpha||^*_{Th} = \sup_{\Sigma} \frac{<\alpha, [\Sigma]>}{2g(\Sigma) - 2}, \qquad (2)$$

where $\alpha \in H^2(M,\mathbb{R})$ and the supremum being taken over all connected, oriented surfaces Σ embedded in M whose genus g is at least 2.

Recall that a *taut* foliation is a codimension one foliation of a closed manifold with the property that every leaf meets a transverse circle. Equivalently, by a result of Dennis Sullivan [2], a codimension one foliation is taut if there exists a Riemannian metric that makes each leaf a minimal surface. Thurston proved that the convex hull of the Euler classes of taut foliations on M is the unit ball for the dual Thurston norm. In particular, the Thurston norm $||e(\mathcal{F})||^*_{Th}$ of the Euler class $e(\mathcal{F}) \in H^2(M,\mathbb{R})$ of a taut foliation $\mathcal{F}$ is no more then one.

We represent the following result.

Theorem 1. *Let M be a closed oriented hyperbolic 3-Manifold and $\mathcal{F}$ be a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature bounded above by the fixed positive constant H_0. Then*

- *If $H_0 \leq 1$, we have $\mathcal{F}$ is taut and $||e(\mathcal{F})||^*_{Th} = 1$.*
- *If $H_0 > 1$, we have*

$$||e(\mathcal{F})||^*_{Th} \leq 2\pi \frac{1600 H_0^2 Vol(M)^2}{C_0^3 inj(M)} + \frac{300 Vol(M)}{inj(M)} + 1,$$

where $C_0 = 2\min\{inj(M), (\coth)^{-1}(H_0)\}$, $Vol(M)$ is the volume of M and $inj(M)$ is the injectivity radius of M.

References

[1] W.P. Thurston, A norm for the homology of 3-manifolds. *Memoirs of the American Mathematical Society*, 59 (339): 99 –130, 1986.

[2] D. Sullivan, A homological characterization of foliations consisting of minimal surfaces, *Comm. Math. Helv.*, 54: 218-223, 1979.