Nijenhuis geometry and its applications

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This presentation is focused on some results of the long-term research programme *Nijenhuis Geometry* initiated several years ago in cooperation with Vladimir Matveev and Andrey Konyaev.

A *Nijenhuis operator* $L = \big(L^i_j(x)\big)$ is defined to be a field of endomorphisms on a smooth manifold M such that its Nijenhuis torsion identically vanishes, i.e.,

$$\mathcal{N}_L(\xi, w) = L^2[\xi, \eta] + [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta] = 0, \tag{1}$$

for arbitrary vector fields ξ, η on M. The pair (M, L) is called a *Nijenhuis manifold*.

Relation (1) is the simplest differential-geometric condition on a field of endomorphisms, and that is the reason why Nijenhuis operators appear in many areas of differential geometry and mathematical physics. In the theory of integrable bi-Hamiltonian systems, they serve as recursion operators and their role in this area has been well understood for many years due to pioneering works by F. Magri, Y. Kosmann-Schwarzbach and F. Turiel. A classical fact in complex geometry is that an almost complex structure is integrable if and only if it is Nijenhuis (Newlander–Nireberg theorem). In the context of metric projective geometry, Nijenhuis operators played a crucial role in various classification problems (AB and V. Matveev). They naturally occur in the study of infinite dimensional Poisson brackets of hydrodynamic type (E. Ferapontov *et al*). Even in algebra, Nijenhuis operators turns out to be useful in the theory of integrable systems on Lie algebras and Lie pencils (A. Panasyuk), and also appear as left symmetric algebras.

Besides various applications, our motivation is as follows. Classical geometries are defined by means of a tensor of order 2. For Riemannian, sub-Riemannian, symplectic and Poisson structures, this tensor is a bilinear form (co- or contravariant, symmetric or skew-symmetric). In this list, one type of tensors is still missing: linear operators. Nijenhuis geometry would be a very natural candidate to fill this gap.

Thus, *Nijenhuis Geometry* research programme is aimed at systematic development of the theory of Nijenhuis manifolds. Our vision and first results are presented in [1–8]. More specifically, our goal is to *re-direct the research agenda* in this area from *tensor analysis at generic points* to studying *singularities and global properties*. The ultimate goal of our research programme is to answer three fundamental questions:

(A) **Local description:** to what form can one bring a Nijenhuis operator near almost every point by a local coordinate change?

(B) **Singular points:** what does it mean for a point to be generic or singular in the context of Nijenhuis geometry? What singularities are non-degenerate/stable? How do Nijenhuis operators behave near non-degenerate and stable singular points?

(C) **Global properties:** what restrictions on a Nijenhuis operator are imposed by the topology of the underlying manifold? And conversely, what are topological obstructions to a Nijenhuis manifold carrying a Nijenhuis operator with specific properties?

Below are some of our easy-to-formulate results in the area.

Theorem 1. *Let L be a Nijenhuis operator and $\sigma_1, \dots, \sigma_n$ be the coefficients of its characteristic polynomial $\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n - \sum_{k=1}^n \sigma_k\, t^{n-k}$. Then in any local coordinate system $x_1, \dots, x_n$*

the following matrix relation hold:

$$J(x)\,L(x) = S_\chi(x)\,J(x), \quad \text{where } S_\chi(x) = \begin{pmatrix} \sigma_1(x) & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1}(x) & \vdots & \ddots & 1 \\ \sigma_n(x) & 0 & \dots & 0 \end{pmatrix} \tag{2}$$

and $J(x)$ is the Jacobi matrix of the collection of functions $\sigma_1, \dots, \sigma_n$ w.r.t. the variables $x_1, \dots, x_n$.

Theorem 2. *Let L be a real-analytic Nijenhuis operator of the form*

$$L(x) = L_{\text{lin}}(x) + R(x), \quad \text{where } L_{\text{lin}}(x) = \operatorname{diag}(x_1, x_2, \dots, x_n)$$

and $R(x)$ denotes a non-linear perturbation (of order ≥ 2). Then $L(x)$ is linearisable, i.e., there exists a real analytic change of variables $x \mapsto y$ such that in the new coordinates $L(y) = \operatorname{diag}(y_1, y_2, \dots, y_n)$.

Theorem 3. *A Nijenhuis operator on a closed connected manifold cannot have non-constant complex eigenvalues.*

Theorem 4. *Consider a real analytic* gl*-regular Nijenhuis operator L (*gl*-regularity means that each eigenvalue of L may have arbitarary multiplicity but only one linearly independent eigenvector). Then there exist local coordinate systems $u = (u^1, \dots, u^n)$ and $v = (v^1, \dots, v^n)$ in which L reduces to the first and second companion forms:*

$$L(u) = L_{\mathsf{comp1}} = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad L(v) = L_{\mathsf{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

where σ_i are the coefficients of the characteristic polynomial of L in the corresponding coordinate system.

Theorem 5. *Let M^2 be either a sphere or a closed Riemann surface of genus ≥ 2. Then M^2 cannot carry any* gl*-regular Nijenhuis operator L except for $L = \alpha \operatorname{Id} + \beta A$, where A is a complex structure on M^2 and $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. A non-orientable closed 2-manifold different from a Klein bottle cannot carry any* gl*-regular Nijenhuis operator.*

References

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