Algebraic and geometric methods in Relativistic Quantum Mechanics and Schwartz distribution spaces defined on Minkowski space-time

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The intent of the general analysis conducted during this talk is to start from Laurent Schwartz distribution spaces, based also on the Minkowski space-time, following the spirit of Linear Algebra and Geometry, and offer precise meanings and rigorous support to many calculus methods of Quantum Mechanics.

Our approach not only provides a rigorous and efficient justification for the use of many quantum mechanics mathematical tools substantially as they usually show up in the physical practice, but - by a "correct" formulation of the calculus methods in terms of contemporary mathematics - it helps to reach a deeper comprehension of the physical structures studied in Quantum Mechanics.

In particular, in this direction, we consider a new definition of state spaces for quantum systems. These structures are often identified with separable Hilbert spaces, leading immediately to the so called "non-normalizable" states - which revealed fundamental in the development of the quantum mechanics. Indeed, those particular states should be normalizable, but with respect other convenient scalar products, orthogonal to the initial one.

Some researchers in theoretical quantum mechanics already understand that the state space of a Quantum system should be a larger structure than Hilbert spaces, sometimes they call it "physical Hilbert space", without introducing a clear definition. We have already shown in the past ([1, 2, 3]) that "physical Hilbert spaces" can be smoothly identified with distribution spaces on suitable Euclidean spaces, depending from the nature of the quantum system considered. Such distribution spaces should be endowed with some algebraic-topological structures, such as the operations of continuous superposition and extended Dirac products.

In particular, the extended Dirac product allows us to introduce new usual scalar products upon some distinguished subspaces of distribution spaces, endowing those subspaces with non-separable Hilbert structures, which clarify definitely the role of the so-called non-normalizable states. For example, the singular Dirac distributions and the celebrated De Broglie waves become elements of those new non-separable Hilbert spaces and, consequently, they acquire the status of normalizable states, as it seems completely natural because of the physical usual probability interpretation of such states. The new operation of continuous-superposition revealed the right tool which allows us to build - in a mathematically rigorous way - the extended Linear Algebra of Dirac, in distribution spaces, using the Schwartz natural topological-linear structures of those spaces.

More precisely, we saw that the natural algebraic-topological structure of those spaces allows to define an extension of the finite linear combination, when the sets indexing the families of vectors are continuous sets, even in the case in which the systems of coefficients show a continuous-infinity of terms different from zero. Now, it appears utterly clear how our approach to distribution theory and QM induces a renewed geometrical vision of the functional analysis developments of the two theories themselves: we realize the existence of continuous bases of distributions and corresponding coordinate systems, continuous matrices of continuous operators, tangent and cotangent spaces of infinite dimensional manifolds, modeled upon distribution spaces, endowed with suitable and comfortable continuous bases, leading to a infinite dimensional differential geometry theory much closer to the finite dimensional one.

As a possible application, we solve the problem of quantizing the relativistic Hamiltonian of a free massive particle massive particle (rest mass different from 0). In distribution state spaces, we find a natural way to define the relativistic Hamiltonian operator and its associated Schrödinger equation. We, then, deduce the equivalent continuity equation for the Born probability density and study some its different (but equivalent) expressions. We determine the possible probability currents and flux velocity fields associated with the particle-field evolution.

The principal properties of the Hamiltonian operator are presented in the following theorem.

Theorem 1. *The relativistic Hamiltonian operator $\hat{H}$ reveals the unique linear continuous operator on $\mathcal{S}'(\mathbb{M}_4)$ sending each de Broglie wave β_p to the tempered distribution $H_{\boldsymbol{p}}\beta_p$, where $\boldsymbol{p}$ denotes the spatial part of p. Moreover, we see that:*

- *$\hat{H}$ reveals Schwartz diagonalizable (Schwartz non-defective): there exists a Schwartz basis of $\mathcal{S}'(\mathbb{M}_4)$ constituted by eigenvectors of $\hat{H}$;*
- *the operator $\hat{H}$ reveals regular and Hermitian in the Schwartz sense: it could be restricted to an endomorphism of the test function space $\mathcal{S}(\mathbb{M}_4)$ and its restriction reveals Hermitian with respect to the standard Dirac inner product of $\mathcal{S}(\mathbb{M}_4)$.*

In non-relativistic QM, the evolution equation takes also the time-dependent form

$$\mathcal{E}'_{\mathbf{H}} \ : \ \imath\hbar\, \psi'(t) = \hat{\mathbf{H}}\psi(t),$$

with

$$\psi : \mathbb{T} \to \mathcal{S}'(\mathbb{X}_3),$$

smooth curve parametrized by time. We desire to find an analogous expression in the relativistic case.

Theorem 2. *Let us fix any $\psi_0 \in \mathcal{S}'(\mathbb{P}_3)$. Set now*

$$\psi(t) = e^{-(\imath/\hbar)t\,\hat{\mathbf{H}}}\,\psi_0,$$

for every time t. Here, as usual in Schwartz linear algebra, we define

$$e^{-(\imath/\hbar)t\,\hat{\mathbf{H}}}\,\psi_0 := \int_{\mathbb{P}_3} e^{-(\imath/\hbar)t\,\mathbf{H}}\,(\psi_0)_\eta\,\eta,$$

for every tempered wave ψ_0 defined upon $\mathbb{X}_3$. Then, the above curve ψ verifies the Schrödinger equation $\mathcal{E}_{\mathbf{H}}$, that is, it fulfills the relation

$$\mathcal{E}_{\mathbf{H}} \ : \ \imath\hbar\, \psi'(t) = \hat{\mathbf{H}}\psi(t),$$

for every time t.

We moreover prove the following theorem.

Theorem 3. *Any solution κ of the Schrodinger equation $\mathcal{E}$ determines a distribution-curve*

$$\psi : \mathcal{S}(\mathbb{T}) \to \mathcal{S}'(\mathbb{X}_3)$$

defined by

$$< \psi(\phi_0), \phi > = < \kappa, \phi_0 \otimes \phi > .$$

Viceversa, any distribution curve, satisfying $\mathcal{E}'$, determines, by the Schwartz kernel theorem, a solution of the Shrodinger equation $\mathcal{E}$.

References

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