Hurwitz Zeta Functions and Ramanujan's Identity for Odd Zeta Values

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One of the famous identities given by Ramanujan which has attracted the attention of several mathematicians over the years is the following intriguing identity involving the odd values of the Riemann zeta function:

Theorem 1 (Ramanujan's formula for $\zeta(2n+1)$). *If α and β are positive real numbers such that $\alpha\beta = \pi^2$ and if $n \in \mathbb{Z} \setminus \{0\}$, then we have*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{m^{-2n-1}}{e^{2\alpha m}-1}\right\}-(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{m^{-2n-1}}{e^{2\beta m}-1}\right\}$$
$$=2^{2n}\sum_{k=0}^{n+1}\frac{(-1)^{k-1}B_{2k}B_{2n-2k+2}}{(2k)!\,(2n-2k+2)!}\alpha^{n-k+1}\beta^{k}. \tag{1}$$

where B_n denotes the n-th Bernoulli number.

Theorem 1 appears as Entry 21 in Chapter 14 of Ramanujan's second notebook [1, 173]. Notice that the function $\frac{1}{e^{2\pi x}-1}$ appears in several of Ramanujan's identities and has the following integral representation:

$$\frac{1}{e^{2\pi x}-1}=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s)}{2\cos\left(\frac{\pi s}{2}\right)}x^{-s}\mathrm{d}s,$$

where (c) denotes the vertical line $\Re(s) = c$ with c an arbitrary real number such that $1 < c < 2$. We generalize this function and define the *Hurwitz kernel* by

$$\Psi(x,a;k):=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}x^{-s}\mathrm{d}s$$
$$=\frac{2a-1}{2\pi x}-\frac{1}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{x^{2k-1}}{x^{2k}+(n+a)^{2k}}, \tag{2}$$

where $\zeta(s,a)$ is the Hurwitz zeta function. Let $\Psi_\alpha(x,a;k) = \Psi\left(\frac{\alpha x}{\pi},a;k\right)$. We now find several identities involving this kernel which generalize Ramanujan's identity such as the following:

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha\beta = \pi^2$ and let $k, N \in \mathbb{N}$. Then, we have*

$$\beta^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\alpha(n+b,a;k)}{(n+b)^{2k(N+1)-1}}+\frac{\zeta(2k(N+1)-1,b)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)$$
$$=(-1)^N\alpha^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\beta(n+a,b;k)}{(n+a)^{2k(N+1)-1}}+\frac{\zeta(2k(N+1)-1,a)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)$$
$$+\sum_{p=0}^{N+1}(-1)^{p+1}\zeta(2kp,a)\,\zeta(2k(N-p+1),b)\,\alpha^{kp-1}\beta^{k(N+1-p)-1}. \tag{3}$$

References

[1] B. C. BERNDT, *Ramanujan's notebooks*, Part II, Springer, New York, 1989