Global asymptotic stability of generalized homogeneous dynamical systems

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The equivalence between uniform asymptotic stability and exponential stability for generalized homogeneous non-autonomous differential equations

$$x' = f(t, x) \tag{1}$$

is established. This results we prove in the framework of general non-autonomous (cocycle) dynamical systems.

Let $\mathbb{R} := (-\infty, +\infty)$ and $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be the space of all continuous functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ equipped with the compact-open topology. Denote by $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, i.e., $\sigma(\tau, f) := f^\tau$ and $f^\tau(t, x) := f(t + \tau, x)$ for any $t, \tau \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Along with equation (1) we consider its *H-class* [4, 2, 6, 10], i.e., the family of equations

$$v' = g(t, v), \tag{2}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{R}\}}$, $f^\tau(t, u) = f(t+\tau, u)$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. We will suppose also that the function f is *regular* [9, ChIV], i.e., for every equation (2) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(t, v, g)$ the solution of equation (2), passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$.

Let $\mathbb{R}^n$ with euclidian norm $|x| := \sqrt{x_1^2 + \ldots + x_n^2}$. Denote by

$$|x|_{r,p} := \left(\Sigma_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}, \tag{3}$$

where $r := (r_1, \ldots, r_n)$, $r_i > 0$ for any $i = 1, \ldots, n$ and $p \geq 1$. Denote by $\rho(x) := |x|_{r,p}$ and $\Lambda_\varepsilon^r := diag(\varepsilon^{r_i})_{i=1}^n$.

Definition 1. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be:

(1) r-homogeneous ($r \in (0, +\infty)^n$) of degree $m \in \mathbb{R}$ [7, 11] if $f(t, \Lambda_\varepsilon^r x) = \varepsilon^m \Lambda_\varepsilon^r f(t, x)$ for any $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$;
(2) Lagrange stable [4] if the set $H(f)$ is compact in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

Remark 2. If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of degree $m \geq 0$, then $f(t, 0) = 0$ for any $t \in \mathbb{R}$.

Definition 3. The trivial solution of equation (1) is said to be:

(1) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f^\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;
(2) *attracting (respectively, uniformly attracting)*, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, x, f^\tau)| = 0$$

for all (respectively, uniformly with respect to) $|x| \leq a$ and $\tau \in \mathbb{R}_+$;
(3) *asymptotically stable (respectively, uniformly asymptotically stable*, if it is uniformly stable and attracting (respectively, uniformly attracting).

Remark 4. 1. Note that from the results given in the works [1],[9] it follows the equivalence of standard definition of uniform stability (respectively, global uniform asymptotically stability) and of the one given above for the equation (1) with regular right hand side.

2. From the results of G. Sell [9] it follows that for the differential equations (1) with the regular and Lagrange stable right hand site f the following statements are equivalent:

(1) the trivial solution of equation (1) is uniformly asymptotically stable;
(2) the trivial motion of the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by (1) [4, Ch.I] is uniformly asymptotically stable.

Theorem 5. *Assume that the function f is r homogeneous of degree zero and Lagrange stable. Then the following statements are equivalent:*

(1) *the trivial solution of equation* (1) *is uniformly asymptotically stable;*
(2) *the trivial solution of equation (1) is globally uniformly asymptotically stable;*
(3) *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,g)) \le \mathcal{N} e^{-\nu t}\rho(u) \quad (4)$$

for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \ge 0$, where $\rho(u) = |u|_{r,p}$.

Remark 6. 1. If the function f is τ-periodic, then the equivalence of the conditions (ii) and (iii) was established in the work [8].

2. If the function f is homogeneous of degree zero (in the classical sense, i.e., $f(t, \varepsilon x) = \varepsilon f(t,x)$ for any $\varepsilon > 0$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^n$), then the equivalence of the uniform asymptotically stability and exponential stability was established in the work [5, Ch.I] (for finite-dimensional case) and in the work [3] (for infinite-dimensional case).

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