Hyper-holomorphically projective mappings of hyper-Kähler manifolds

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Definition 1. [2, 3] *A hyper-Kähler manifold* is a Riemannian manifold (M^n, g, F, G, H) with three covariant constant orthogonal automorphisms F, G, H of the tangent bundle which satisfy the quaternionic identities

$$F^2 = G^2 = G^2 = FGH = -I.$$

The symbol I denotes the identity tensor of type $(1,1)$ in the manifold. In terms of a local coordinate system we might write:

$$F^h_\alpha F^\alpha_i = -\delta^h_i, \qquad G^h_\alpha G^\alpha_i = -\delta^h_i, \qquad H^h_\alpha H^\alpha_i = -\delta^h_i, \tag{1}$$

$$G^h_\alpha H^\alpha_i = -H^h_\alpha G^\alpha_i = F^h_i, \qquad H^h_\alpha F^\alpha_i = -F^h_\alpha H^\alpha_i = G^h_i, \qquad F^h_\alpha G^\alpha_i = -G^h_\alpha F^\alpha_i = H^h_i, \tag{2}$$

$$\nabla_j F^h_i = 0, \qquad \nabla_j G^h_i = 0, \qquad \nabla_j H^h_i = 0, \tag{3}$$

Obviously, the covariant derivative in (3) is compatible with the Riemannian metric g. In this case the metric is referred to as a hyper-Kähler one.

In a hyper-Kähler manifold, let us consider a curve $x(t)$ satisfying differential equations:

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)F^h_i\frac{dx^i}{dt} + \gamma(t)G^h_i\frac{dx^i}{dt} + \delta(t)H^h_i\frac{dx^i}{dt},$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are certain functions of the parameter t, the symbol Γ^h_{ij} denotes connection compatible with the Riemannian metric g. We call such a curve a *hyper-holomorphically planar curve (HHP-curve)*. The HHP-curves are a generalization of holomorphically planar curves [1].

Suppose two hyper-Kähler manifolds (M^n, g, F, G, H) and $(\overline{M}^n, \overline{g}, F, G, H)$ are given and the defined triple of the affinors F, G, H is the same in both manifolds.

A mapping $\pi : (M^n, g, F, G, H) \to (\overline{M}^n, \overline{g}, F, G, H)$ is an *hyper-holomorphically projective mapping (HHP-mapping)* if any HHP-curve of (M^n, g, F, G, H) is mapped under π onto an HHP-curve in $(\overline{M}^n, \overline{g}, F, G, H)$.

Theorem 2. *If two hyper-Kähler manifolds* (M^n, g, F, G, H) *and* $(\overline{M}^n, \overline{g}, F, G, H)$ *are in hyper-holomorphically projective correspondence, then their Levi-Civita connections related to each other as*

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_\alpha F^\alpha_{(i}F^h_{j)} - \psi_\alpha G^\alpha_{(i}G^h_{j)} - \psi_\alpha H^\alpha_{(i}H^h_{j)},$$

where ψ_i *is some gradient vector.*

Theorem 3. *Let a hyper-Kähler manifold* (M^n, g, F, G, H) *admit* HHP*-mappings. Then the object*

$$\overline{T}^h_{ij} = \Gamma^h_{ij} - \frac{1}{n+4}\big(\Gamma^\alpha_{\alpha(i}\delta^h_{j)} - \Gamma^\alpha_{\alpha\beta}F^\beta_{(i}F^h_{j)} - \Gamma^\alpha_{\alpha\beta}G^\beta_{(i}G^h_{j)} - \Gamma^\alpha_{\alpha\beta}H^\beta_{(i}H^h_{j)}\big)$$

is invariant under any HHP*-mapping.*

References

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[3] Hitchin N. Hyper-Kähler manifolds. *Séminaire Bourbaki, Astérisque* 1992, vol. 206, No. 748, 3, p. 137-166.