Gottlieb groups of some Moore spaces

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In this work, we present some computations of Gottlieb groups of Moore spaces $M(A,n)$ for some classes of finitely generated abelian groups A.

Given $m \geq 1$, recall that the m-th *Gottlieb group* $G_m(X)$ of a space X has been defined in [4, 5] as the subgroup of the homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $f\colon \mathbb{S}^m \to X$ such that $f \vee \iota_X \colon \mathbb{S}^m \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^m \times X \to X$. Notice that $\alpha \in G_m(\Sigma X)$ if and only if the generalized Whitehead product $[\alpha, \iota_{\Sigma X}] = 0$ (see [1, Proposition 5.1]).

First, we recall from [5, Theorems 5.2 and 5.4]:

Theorem 1. *Let A be a finitely generated abelian group and $n \geqslant 3$. Then,*

$$G_n(M(A,n)) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd and } \mathrm{rk}(A) \neq 1, \\ 2\mathbb{Z} \subseteq \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n \neq 1,3,7 \text{ is odd and } A = \mathbb{Z}, \\ \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n = 1,3,7 \text{ and } A = \mathbb{Z} \end{cases}$$

We point out that the result above has been stated also in [2] for $n \geqslant 3$. In addition, [2, Corollary 4.4] claims that if n is odd, then $G_n(M(\mathbb{Z} \oplus T, n))$ is infinite cyclic, where T is a finite abelian group.

As stated in [2, Remark 4.5], it would be interesting to compute other Gottlieb groups for some Moore spaces, such as $G_{n+1}(M(A,n))$. We will do this for a finitely generated abelian group A which its torsion subgroup has order $2 \pmod 4$. We notice that on [3, Chapter 3] there are some results on $G_{n+1}(M(A,n))$ only for A having torsion subgroup with odd order.

Our main result is:

Theorem 2. *Let A be a finite abelian group with order $|A| \equiv 2 \pmod 4$. Then $G_{n+1}(M(\mathbb{Z} \oplus A, n)) = 0$, for $n \geq 3$, and $G_{n+2}(M(\mathbb{Z} \oplus A, n)) = 0$, for $n \geq 4$.*

Furthermore, investigations of $G_{n+k}(M(\mathbb{Z} \oplus A, n))$ for $k = 3,4,5$ and A as above, is planned as well.

References

[1] Martin Arkowitz. The generalized Whitehead product. *Pacific J. Math.* 12 : 7–23, 1962.
[2] Martin Arkowitz and Ken-ichi Maruyama. The Gottlieb group of a wedge of suspensions. *J. Math. Soc. Japan*, 66(3) : 735–743, 2014.
[3] Marek Golasiński and Juno Mukai. *Gottlieb and Whitehead center groups of spheres, projective and Moore spaces*, Springer, 2014.
[4] Daniel H Gottlieb. A certain subgroup of the fundamental group. *American Journal of Mathematics*, 87(4) : 840–856, 1965.
[5] Daniel H Gottlieb. Evaluation subgroups of homotopy groups. *American Journal of Mathematics*, 91(3) : 729–756, 1969.