Inner semi-continuity of medial axes and conflict sets

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A central notion in pattern recognition is that of the *medial axis* M_X of a closed, nonempty, proper subset $X \subset \mathbb{R}^n$. Namely, M_X consists of all those points $a \in \mathbb{R}^n$ for which there is more than one closest point (with respect to the Euclidean distance $d(a, X)$) in X:

$$M_X := \{a \in \mathbb{R}^n \mid \#m(a) > 1\} \text{ where } m(a) := \{x \in \mathbb{R}^n \mid ||a - x|| = d(a, X)\}.$$

The definition goes back to H. Blum (cf. [3]) who gave it for $X = \partial D$ where $D \subset \mathbb{R}^n$ is a bounded domain. Then, knowing the 'skeleton' $M_X \cap D$ and $d(\cdot, X)|_{M_X}$ ('compressed data') one can reconstruct the 'shape' D.

The medial axis has long been known for being highly unstable (cf. e.g. [4]): the smallest deformation of X may lead to an important change in M_X (think of X as a circle in the plane — M_X is its centre, while the same circle but now with the smallest $\mathscr{C}^\infty$ protuberance yields a medial axis that is a segment). However, this point of view has a flaw — it sees the modification as through a blackbox, there is an initial state and a final one with nothing in between.

Our aim is to provide the right setting for considering the deformation of X which is the *(Painlevé)-Kuratowski convergence of closed sets* and to show in this case the inner-semicontinuity of the medial axis. The most general result we have, and one that turns out ot be optimal already in $\mathbb{R}^n$, can be stated as follows:

Theorem 1. *Let $\mathcal{M}$ be a connected complete Riemannian manifold and Π a T_1 topological space of parameters with a distinguished non-isolated point 0 having a countable basis of neighbourhoods. We write $\Omega_{X,p}$ for the set of geodesics of minimal length connecting a point in $m(p)$ with p and $\gamma_{X,p}$ for such a geodesic originating at p. Assume that $X \subset \Pi \times \mathcal{M}$ has closed t-sections and we have the Kuratowski convergence $X_t \xrightarrow{K} X_0$. Then for $M = \{(t, x) \in \Pi \times \mathcal{M} \mid \exists \gamma_{X_t,p}, \tilde{\gamma}_{X_t,p} \in \Omega_{X_t,p} : \gamma_{X_t,p} \neq \tilde{\gamma}_{X_t,p}\}$, we have*

$$\liminf_{\pi(M) \ni t \to 0} M_t \supset M_0$$

where the lower limit is understood in the Kuratowski sense:

$$x \in \liminf_{\pi(M) \ni t \to 0} M_t \Leftrightarrow \forall \pi(M) \setminus \{0\} \ni t_\nu \to t_0, \exists M_{t_\nu} \ni x_\nu \to x.$$

We will show how this applies in singularity theory in $\mathbb{R}^n$ giving a criterion for M_X to reach certain singularities of X when X is definable in some o-minimal structure (e.g. semi-algebraic), cf. [2].

Finally, we will discuss a counterpart of this theorem in the case of *conflict sets* of finite families of closed, pairwise disjoint sets, instead of the medial axis, cf. [1]. The conflict set of two sets is their set of equidistant points. In case of more than two sets it can be seen as the set of points at which the distance wavefronts emanating from the sets meet.

References

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