The diameter-width-ratio for complete and pseudo-complete sets

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Any set $A \subset \mathbb{R}^n$ fulfilling $A = t - A$ for some $t \in \mathbb{R}^n$ is called symmetric and 0-symmetric if $t = 0$. We denote the family of all (convex) bodies (full-dimensional compact convex sets) by $\mathcal{K}^n$ and the family of 0-symmetric bodies by $\mathcal{K}^n_0$. For any $K \in \mathcal{K}^n$ the gauge function $\|\cdot\|_K : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\|x\|_K = \inf\{\rho > 0 : x \in \rho K\}.$$

In case $K \in \mathcal{K}^n_0$ we see that $\|\cdot\|_K$ defines a norm. However, even for a non-symmetric unit ball K, one may approximate the gauge function by the norms induced from symmetrizations of K

$$\|x\|_{\mathrm{conv}(K\cup(-K))} \leq \|x\|_K \leq \|x\|_{K\cap(-K)}.$$

It is natural to request that $K \cap (-K) = K = \mathrm{conv}(K \cup (-K))$ if K is symmetric, which is true if and only if 0 is the center of symmetry of K. This motivates the definition of a meaningful center for general K. We introduce one of the most common asymmetry measures, which is best suited to our purposes, and choose the center matching it.

The *Minkowski asymmetry* of K (denoted by $s(K)$) is defined as

$$s(K) := \inf\{\rho > 0 : K - c \subset \rho(c - K), \quad c \in \mathbb{R}^n\},$$

and a *Minkowski center* of K is any $c \in \mathbb{R}^n$ such that $K - c \subset s(K)(c - K)$. Moreover, if 0 is a Minkowski center, we say K is *Minkowski centered*. It is well-known that $s(K) \in [1, n]$ for all $K \in \mathcal{K}^n$, with $s(K) = 1$ if and only if K is symmetric and $s(K) = n$ if and only if K is a simplex.

Notice that there always exists some $x \in \mathbb{R}^n$ such that $\alpha(K)\|x\|_{K\cap(-K)} = \|x\|_{\mathrm{conv}(K\cup(-K))}$, which means that we have equality in the complete chain in the equality above for that x if $\alpha(K) = 1$. We investigate the region of all possible values for the parameter $\alpha(K)$ for Minkowski centered $K \in \mathcal{K}^2$ in dependence of the asymmetry of K.

We show that $\alpha(K) \geq \frac{2}{s(K)+1}$ for all Minkowski centered K, and that in the planar case $\alpha(K) = 1$ implies $s(K) \leq \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$ denotes the golden ratio.

We give a complete description of the possible α-values of K in the planar case in dependence of its Minkowski asymmetry. Moreover, we derive the (unique) family of convex bodies that fulfill the upper bound of $\alpha(K)$.

K is called complete (w.r.t. C), if any proper superset of it has a greater diameter than K.

We also present an application on the diagram of the α-values of K for the diameter-width ratio for complete and pseudo-complete sets. We extend the results on the bounds for $\alpha(K)$ and describe the region of all possible values for this parameter for Minkowski centered convex compact set K in dependence of the asymmetry of K.

Theorem 1. *Let K be Minkowski centered. Then*

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2 - 1}\right\}.$$

Moreover, for every pair (α, s), such that $\frac{2}{s+1} \leq \alpha \leq \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered K, such that $s(K) = s$ and $\alpha(K) = \alpha$.

Consider $K \in \mathcal{K}^n$ and $C \in \mathcal{K}_0^n$. For $s \in \mathbb{R}^n \setminus \{0\}$ the s-breadth of K w.r.t. C is the distance between the two parallel supporting hyperplanes of K with normal vector s, i.e.,

$$b_s(K, C) := \frac{\max_{x,y \in K} s^T(x-y)}{\max_{x \in C} s^T x}.$$

The minimal s-breadth

$$w(K, C) := \min_{s \in mathbbR^n \setminus \{0\}} b_s(K, C)$$

and the maximal s-breadth

$$D(K, C) := \max_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K, C)$$

are called width and diameter of K w.r.t. C, respectively.

We present a quantitative result on the diameter-width ratio for for complete sets.

Theorem 2. *Let K, C be convex compact sets and C be 0-symmetric be such that K is complete w.r.t. C. Then*

$$\frac{D(K, C)}{w(K, C)} \leq \frac{s(K) + 1}{2}.$$

Moreover, for $n > 2$ even and for any $s \in [1, n-1]$ there exists $K, C \in \mathcal{K}^n$ such that K is complete w.r.t. C with $s(K) = s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$, while for $n > 2$ odd and any $s \in [1, n]$ there exists $K \in \mathcal{K}^n$ which is complete w.r.t. C with $s(K) = s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$.