On the possibility of joining two pairs of points in convex domains using paths

Dovhopiatyi Oleksandr
(Zhytomyr Ivan Franko State University)
E-mail: `Alexdov1111111@gmail.com`

Recall, that a set C is *convex* if any pair of points $x, y \in C$ may be joined by some segment which belongs to C, as well. We define the Euclidean distance between sets and the Euclidean diameter by the formulae

$$d(A,B) = \inf_{x\in A, y\in B} |x-y|\,, \quad d(A) = \sup_{x,y\in A} |x-y|\,.$$

Sometimes we also write $\mathrm{dist}\,(A,B)$ instead $d(A,B)$ and $\mathrm{diam}\,E$ instead $d(E)$, as well. As usually, we set

$$B(x_0,r) = \{x\in\mathbb{R}^n : |x-x_0|<r\}\,,$$

$$S(x_0,r) = \{x\in\mathbb{R}^n : |x-x_0|=r\}\,.$$

We emphasize that, the results established here have already been obtained in particular case, when a domain is the unit ball [1]. Concerning some applications of modulus inequalities in the mapping theory, see [2], cf. [3]–[4].

Theorem 1. *Let D' be a bounded convex domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $E := B(y_*,\delta_*/2)$ be a ball centered at the point $y_*\in D'$, where $\delta_* := d(y_*,\partial D')$. Let $z_0\in\partial D'$. Then for any points $A,B\in B(z_0,\delta_*/8)\cap D'$ there are points $C,D\in\overline{B(y_*,\delta_*/2)}$, for which the segments $[A,C]$ and $[B,D]$ are such that*

$$\mathrm{dist}\,([A,C],[B,D]) \geqslant C_0\cdot |A-B|\,, \tag{1}$$

where $C_0>0$ is some constant depending only on δ_* and $d(D')$.

Recall that, a Borel function $\rho:\mathbb{R}^n\to[0,\infty]$ is called *an admissible* for a family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1 \tag{2}$$

holds for any locally rectifiable path $\gamma\in\Gamma$. *A modulus* of Γ is defined as follows:

$$M(\Gamma) = \inf_{\rho\in\,\mathrm{adm}\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,. \tag{3}$$

The following statements hold.

Corollary 2. *Let, under conditions of Theorem 1, Γ denotes the family of all paths joining the segments $[A,C]$ and $[B,D]$ in D'. Then*

$$M(\Gamma) \leqslant \frac{m(D')}{C_0^n}\cdot\frac{1}{|A-B|^n}\,, \tag{4}$$

where M is the modulus of families of paths defined in (3), $m(D')$ denotes the Lebesgue measure of D', and C_0 is a constant in (1).

Corollary 3. *Let, under conditions of Theorem 1,* Γ *denotes the family of all paths joining the segments* $[A, C]$ *and* $[B, D]$ *in* D'*. Then*

$$M(\Gamma) \geqslant \widetilde{c_n} \cdot \log\left(1 + \frac{3\delta_*}{8|A-B|}\right)\,, \tag{5}$$

where M *is the modulus of families of paths defined in (3),* $\widetilde{c_n} > 0$ *is some constant depending only on* n *and* D'*.*

References

[1] Sevost'yanov E.A. (2022). On logarithmic Hölder continuity of mappings on the boundary. *Annales Fennici Mathematici, 47,* 251–259.

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[4] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2005). On Q-homeomorphisms. *Ann. Acad. Sci. Fenn. Math., 30*(1), 49–69 (2005).