On geodesic lines of Riemannian metric for Navier-Stokes equations

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Theorem 1. *The 14D Riemann metric in local coordinates* $\vec{x}=(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)$

$$ds^2=2\,dxdu+2\,dydv+2\,dzdw+(-W(\vec{x},t)w-V(\vec{x},t)v-U(\vec{x},t)u)\,dt^2+$$
$$+\left(-U(\vec{x},t)p-u\,(U(\vec{x},t))^2-uP(\vec{x},t)+w\mu\,\frac{\partial}{\partial z}U(\vec{x},t)-wU(\vec{x},t)W(\vec{x},t)\right)d\eta^2+$$
$$+\left(v\mu\,\frac{\partial}{\partial y}U(\vec{x},t)-vU(\vec{x},t)V(\vec{x},t)+u\mu\,\frac{\partial}{\partial x}U(\vec{x},t)\right)d\eta^2+2\,d\eta d\xi+2\,d\rho d\chi+2\,dmdn+$$
$$\left(-V(\vec{x},t)p-vP(\vec{x},t)-v\,(\vec{x},t))^2-V(\vec{x},t)W(\vec{x},t)w+v\mu\,\frac{\partial}{\partial y}V(\vec{x},t)-uU(\vec{x},t)V(\vec{x},t)\right)d\rho^2+$$
$$\left(u\mu\,\frac{\partial}{\partial x}V(\vec{x},t)\right)d\rho^2+\left(-uU(\vec{x},t)W(\vec{x},t)-w\,(W(\vec{x},t))^2-wP(\vec{x},t)+w\mu\,\frac{\partial}{\partial z}W(\vec{x},t)\right)dm^2+$$
$$\left(v\mu\,\frac{\partial}{\partial y}W(\vec{x},t)-vV(\vec{x},t)W(\vec{x},t)+u\mu\,\frac{\partial}{\partial x}W(\vec{x},t)-W(\vec{x},t)p\right)dm^2 \qquad (1)$$

is the Ricci-flat,

$$R_{44}=U_x+V_y+W_z=0,\ R_{55}=0,\ R_{66}=0,\ R_{77}=0$$

on solutions of Navier-Stokes system of equations

$$\frac{\partial}{\partial t}\vec{Q}(\vec{x},t)+(\vec{Q}(\vec{x},t)\cdot\vec{\nabla})\vec{Q}(\vec{x},t)-\mu\,\Delta\vec{Q}(\vec{x},t)+\vec{\nabla}P(\vec{x},t)=0,\quad \vec{\nabla}\cdot\vec{Q}(\vec{x},t)=0, \qquad (2)$$

where $\vec{Q}(\vec{x},t)=[U(\vec{x},t),V(\vec{x},t),W(\vec{x},t)]$ *are the components of velocity and* $P(\vec{x},t)$ *is pressure of liquid.(see e.g. [1-2])*

To obtain the metric (1) presentation the NS-system of equations in the form of laws conservations

$$U_t+(U^2-\mu U_x+P)_x+(UV-\mu U_y)_y+(UW-\mu U_z)_z=0$$
$$V_t+(V^2-\mu V_y+P)_y+(UV-\mu V_x)_x+(VW-\mu V_z)_z=0,$$
$$W_t+(W^2-\mu W_z+P)_z+(UW-\mu W_x)_x+(VW-\mu W_y)_y=0,$$
$$(U_x+V_y+W_z)=0,$$

is used.

The metric (1) belongs to the class of the Riemann spaces with vanishing scalar Invariants. Their geodesics with respect to the coordinates η,ρ,m,ξ,χ,n has form of equations direct lines

$$\ddot{\eta}=0,\ \ddot{\rho}=0,\ \ddot{m}=0,\ \ddot{\xi}=0,\ \ddot{\chi}=0,\ \ddot{n}=0,$$

and in this sense to them the partially-projective spaces of V.Kagan corresponds.

For the coordinates $[x,y,z,t]$ the equations of geodesics of metric (1) are

$$\frac{d^2}{ds^2}x\,(s)=1/2\,\left(\dot{m}(s)\right)^2U\,(x,y,z,t)\,W\,(x,y,z,t)-1/2\,\left(\dot{m}(s)\right)^2\mu\,\frac{\partial}{\partial x}W\,(x,y,z,t)+$$
$$+1/2\,\left(\dot{\eta}(s)\right)^2\left(U\,(x,y,z,t)\right)^2+1/2\,\left(\dot{\rho}\right)^2U\,(x,y,z,t)\,V\,(x,y,z,t)-1/2\,\left(\dot{\rho}\right)^2\mu\,\frac{\partial}{\partial x}V\,(x,y,z,t)-$$
$$-1/2\,\left(\dot{\eta}\right)^2\mu\,\frac{\partial}{\partial x}U\,(x,y,z,t)+1/2\,U\,(x,y,z,t)\left(\frac{d}{ds}t\,(s)\right)^2+1/2\,\left(\dot{\eta}(s)\right)^2P\,(x,y,z,t)\,,$$

$$\frac{d^2}{ds^2}t(s) = 1/2\,W(x,y,z,t)\left(\frac{d}{ds}m(s)\right)^2 + 1/2\,U(x,y,z,t)\left(\frac{d}{ds}\eta(s)\right)^2 + 1/2\,V(x,y,z,t)\left(\frac{d}{ds}\rho(s)\right)^2.$$

$$\frac{d^2}{ds^2}y(s) = ...,\ \frac{d^2}{ds^2}z(s) = ...$$

The equations of geodesics for dual coordinates $[u, v, w, p]$ form the linear system of the second order equations

$$\frac{d^2}{ds^2}u(s) = A_1\,u(s) + B_1\,v(s) + C_1\,w(s) + E_1\,p(s),\quad \frac{d^2}{ds^2}v(s) = A_2\,u(s) + B_2\,v(s) + C_2\,w(s) + E_2\,p(s),$$

$$\frac{d^2}{ds^2}w(s) = A_3\,w(s) + B_3\,v(s) + C_3\,w(s) + E_3\,p(s),\quad \frac{d^2}{ds^2}p(s) = A_4\,u(s) + B_4\,v(s) + C_4\,w(s) + E_4\,p(s),$$

with the coefficients depending on the solutions $U(x, y, z, t), V(x, y, z, t), W(x, y, z, t), P(x, y, z, t$ of the system (2).

On the base of solutions of equations for the Killing vectors of the metric

$$K_{i,j} + K_{j,i} - 2\Gamma^k_{ij}K_k = 0, \quad or \quad K^k g_{ij,k} + g_{ik}K^k, j + g_{jk}K^k, i = 0, \tag{4}$$

a new examples of reductions and solutions of the system (2) are constructed.

Properties of the Lie derivative for the connection coefficients of the metric (1) and the vector field of the form $u^i = g^i_k v^k$

$$u^i_{j,k} + u^n\Gamma^i_{jk},_n + u^n_{,j}\Gamma^i_{nk} + u^n_{,k}\Gamma^i_{jn} - u^n_{,n}\Gamma^i_{jk} = 0,$$

where Γ^i_{jk}-are the coefficients of connection of the metric (1) with the aim of constructing new examples of solutions to the system (2) are discussed.

Another possibility for studying the properties of the NS system by the geometric method is the use of differential Beltrami parameters of the metric (1) $\Delta_2(f) = g^{ij}\frac{\partial^2 f}{\partial x_i, x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}$. As example, in particular case $f = \psi(x, y, z, t, 0, 0, 0, u, v, w, p, 0, 0, 0)$, from solutions of the linear equation with variable coefficients $\Delta_2(f) = 0$ the relation

$$(U(\vec{x},t) - W(\vec{x},t))P(\vec{x},t)/\mu = U(\vec{x},t)\frac{\partial}{\partial z}U(\vec{x},t) - W(\vec{x},t)\frac{\partial}{\partial x}V(\vec{x},t) -$$

$$-W(\vec{x},t)\frac{\partial}{\partial x}U(\vec{x},t) - W(\vec{x},t)\frac{\partial}{\partial x}W(\vec{x},t) + \frac{\partial}{\partial z}V(\vec{x},t)U(\vec{(x},t) + \frac{\partial}{\partial z}W(\vec{x},t)U(\vec{x},t),$$

between velocity and pressure can be derived and that can be applied to the studying properties of solutions of the system (2).

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References

[1] Dryuma V. S., *The Ricci-flat spaces related to the Navier-Stokes equations*, Buletinul Academiei de Stiinte a Republicii Moldova., Matematica, 2(69), 2012. P. 99–102.

[2] Dryuma V. S., *The Riemann and Einstein geometries in the theory of ODE's, their applications and all that*, New Trends in Integrability and Partial Solvability., Kluwer Publisher (A.B. Shabat at al.(eds.), 2004. P.115–156.

[3] Dryuma V. S., *Geometric approach to the study of the Navier-Stokes equations*, Internationa Conference "Modern Achievements in Symmetries of Differential Equations (Symmetry 2022), December 13-16, 2022, P.39–41.Suranaree University of Technology, Thailand, `http//math/sut.ac.th/conference/`

[4] Dryuma V. S., *On polynomial solutions of the Navier-Stokes and Euler equations*, Internatiomnal Conference "Lobachevsky Readings", for the IMC"'22, Collection of Proceedings,(Kazan, June 30 - July 4), v.62, 2022, P.39–41.