On controllability problems for the heat equation in a half-plane in the case of a pointwise control in the Dirichlet boundary condition

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Consider the following control system in a half-plane

$$w_t = \Delta w, \qquad x_1 > 0,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1}$$

$$w\big(0, (\cdot)_{[2]}, t\big) = \delta_{[2]} u(t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{2}$$

$$w\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = w^0, \qquad x_1 > 0,\ x_2 \in \mathbb{R}, \tag{3}$$

where $T > 0$, $u \in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m = 1,2$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. The subscripts [1] and [2] associate with the variable numbers, e,g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively.

Let $\mathbb{R}_+ = (0, +\infty)$. Consider the following spaces of Sobolev type

$$H^s_{⓪} = \left\{ \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \,\middle|\, \left(\forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2 \ \left(\alpha_1 + \alpha_2 \le s \Rightarrow \frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in L^2(\mathbb{R}_+ \times \mathbb{R}) \right) \right) \wedge \left(\forall k = \overline{0, s-1} \ \frac{\partial^k \varphi(0^+, (\cdot)_{[2]})}{\partial x_1^k} = 0 \right) \right\}, \quad s = \overline{0,3},$$

with the norm

$$\|\varphi\|^s_{⓪} = \left(\sum_{\alpha_1+\alpha_2 \le s} \left(\left\| \frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in H^s_{⓪}, \quad s = \overline{0,3},$$

and $H^{-s}_{⓪} = \left(H^s_{⓪}\right)^*$ with the strong norm $\|\cdot\|^{-s}_{⓪}$ of the adjoint space. We have $H^0_{⓪} = L^2(\mathbb{R}_+ \times \mathbb{R})$.

We consider control system (1)–(3) in $H^{-l}_{⓪}$, $l = \overline{1,3}$, i.e. $\left(\frac{d}{dt}\right)^s w : [0,T] \to H^{-1-2s}_{⓪}$, $s = 0,1$, $w^0 \in H^{-1}_{⓪}$. We treat equality (2) as the value of the distribution w at $x_1 = 0$ (see the definition of a distribution's value at a point [1, Chap. 1] and the definition of a distribution's value at a line [2]).

Definition 1. A state $w^0 \in H^{-1}_{⓪}$ is said to be controllable to a target state $w^T \in H^{-1}_{⓪}$ in a given time $T > 0$ if there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution w to system (1)–(3) and $w\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) = w^T$.

Definition 2. A state $w^0 \in H^{-1}_{⓪}$ is said to be approximately controllable to a target state $w^T \in H^{-1}_{⓪}$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon \in L^\infty(0,T)$ such that there exists a unique solution w_ε to system (1)–(3) with $u = u_\varepsilon$ and $\left\| w_\varepsilon\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) - w^T \right\|^{-1}_{⓪} < \varepsilon$.

The main goal of the paper is to study whether the state w^0 is controllable (approximately controllable) to a target state w^T in the time T.

Note that controllability problems for the heat equation in domains bounded with respect to spatial variables were investigated rather completely in a number of papers. However, these problems for the heat equation in domains unbounded with respect to spatial variables have not been fully studied.

For control system (1)–(3), the following assertions are obtained in a given time $T > 0$ under the control bounded by a given constant ($|u(t)| \le U$, $t \in [0, T]$): a necessary condition for controllability from the origin; necessary and sufficient conditions for controllability; sufficient conditions for approximate controllability in terms of Markov power moment problem constructed according to the control problem data.

Using the generalised Laguerre polynomials, we also construct orthogonal bases in special spaces of Sobolev type. With the aid of the constructed bases, we obtain necessary and sufficient conditions for approximate controllability in a given time for system (1)–(3) in the case of L^∞-control. The results are illustrated by an example:

Example 3. Let $T = 1/2$,

$$w^0(x) = \frac{x_1}{T^2} e^{-\frac{|x|^2}{4T}}, \quad w^T(x) = \frac{x_1}{8T^2} e^{-\frac{|x|^2}{8T}}, \quad x_1 > 0,\ x_2 \in \mathbb{R}.$$

Verifying the obtained necessary and sufficient conditions for approximate controllability in a given time for system (1)–(3), we conclude that the state w^0 is approximately controllable to the state w^T in the time $T = 1/2$. Using the algorithm given in [3], we construct end states $w_l^N(\cdot, T) \in H_{\textcircled{0}}^{-1}$ and piecewise constant controls $u_{N,l}$ depending on two parameters N and l, $l = \overline{2(N+2), \infty}$, $N = \overline{1, \infty}$, such that

$$\left\| w_l^N(\cdot, T) - w^T \right\|_{\textcircled{0}}^{-1} \to 0, \quad \text{as } N \to \infty,\ l \to \infty.$$

All obtained results have been published in [3].

References

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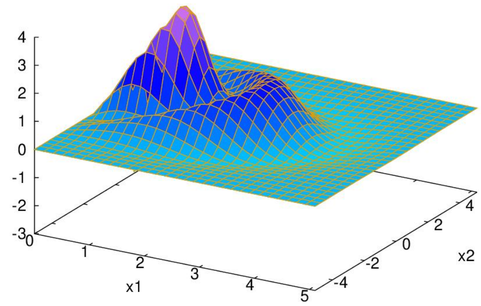

4cm
(a) t

Figure 1. $w_l^N(\cdot, T) - w^T$, $N = 3$, $l = 50$.

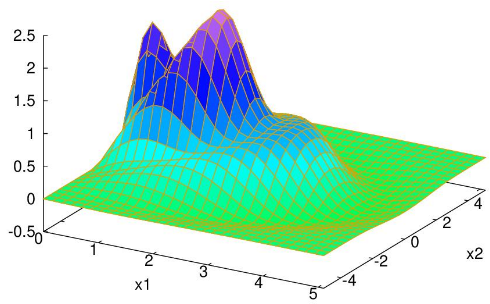

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Figure 2. $w_l^N(\cdot, T) - w^T$, $N = 4$, $l = 200$.

Figure 3. The influence of the controls $u_{N,l}$ on the difference $w_l^N(\cdot, T) - w^T$.