On direct limits of Minkowski's balls, domains, and their critical lattices

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We construct direct systems of Minkowski, Davis and Chebyshev-Cohn balls and domains, direct systems of their critical lattices and calculate their direct limits. By (general) Minkowski balls we mean (two-dimensional) balls in $\mathbb{R}^2$ of the form

$$D_p: \ |x|^p + |y|^p \leq 1, \ p \geq 1. \tag{1}$$

From the proof of Minkowski's conjecture [1, 2, 3, 4, 5, 8] in notations [8, 9] we have next expressions for critical determinants and their lattices:

Theorem 1. (1) $\Delta(D_p) = \Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p, \ 2 \leq p \leq p_0$;
(2) $\sigma_p = (2^p - 1)^{1/p}$,
(3) $\Delta(D_p) = \Delta_p^{(1)} = \Delta(p, 1) = 4^{-\frac{1}{p}} \frac{1+\tau_p}{1-\tau_p}, \ 1 \leq p \leq 2, \ p \geq p_0$,
(4) $2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p < 1$,
here p_0 is a real number that is defined unique by conditions $\Delta(p_0, \sigma_p) = \Delta(p_0, 1), \ 2,57 < p_0 < 2,58, \ p_0 \approx 2.5725$
For their critical lattices respectively $\Lambda_p^{(0)}, \ \Lambda_p^{(1)}$ next conditions satisfy: $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ are two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p with the property that $(1,0) \in \Lambda_p^{(0)}, \ (-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$,

Denote by $V(D_p)$ the volume (area) of D_p.

Proposition 2. *The volume of Minkowski ball D_p is equal $4\frac{(\Gamma(1+\frac{1}{p}))^2}{\Gamma(1+\frac{2}{p})}$.*

Proof. (by Minkowski). Let $x^P + y^p \leq 1, x \geq 0, y \geq 0$. Put $x^p = \xi, y^p = \eta$.

$$V(D_p) = \frac{4}{p^2} \iint \xi^{\frac{1}{p}-1} \eta^{\frac{1}{p}-1} d\xi d\eta, \tag{2}$$

where the integral extends to the area

$$\xi + \eta \leq 1, \ \xi \geq 0, \ \eta \geq 0.$$

Expression (2) can be represented in terms of Gamma functions, and we get

$$V(D_p) = 4\frac{(\Gamma(1+\frac{1}{p}))^2}{\Gamma(1+\frac{2}{p})}.$$

□

We consider balls of the form

$$D_p: \ |x|^p + |y|^p \leq 1, \ p \geq 1,$$

and call such balls with $1 < p < 2$ *Minkowski balls*. Continuing this, we consider the following classes of balls and circles.

- *Davis balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Chebyshev-Cohn balls*: $|x|^p + |y|^p \leq 1$ for $p \geq p_0$;

Let D be a fixed bounded symmetric about origin convex body (*centrally symmetric convex body* for short) with volume $V(D)$.

Proposition 3. [6]. *If D is symmetric about the origin and convex, then $2D$ is convex and symmetric about the origin.*

Corollary 4. *Let m be integer $m \geq 0$ and n be natural greater m. If $2^m D$ centrally symmetric convex body then $2^n D$ is again centrally symmetrc convex body.*

Proof. Induction.
We consider the following classes of balls (see above) and domains.

- *Minkowski domains*: $2^m D_p$, integer $m \geq 1$, for $1 \leq p < 2$;
- *Davis domains*: $2^m D_p$, integer $m \geq 1$, for $p_0 > p \geq 2$;
- *Chebyshev-Cohn domains*: $2^m D_p$, integer $m \geq 1$, for $p \geq p_0$;

Proposition 5. *Let m be integer, $m \geq 1$. If Λ is the critical lattice of the ball D_p than the sublattice Λ_{2^m} of index 2^m is the critical lattice of the domain $2^{m-1} D_p$.*

The direct system of Minkowski balls and domains has the form (3), where the multiplication by 2 is the continuous mapping

$$D_p \xrightarrow{2} 2D_p \xrightarrow{2} 2^2 D_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m D_p \xrightarrow{2} \cdots \tag{3}$$

The direct system of critical lattices has the form (4), where the multiplication by 2 is the homomorphism of abelian groups

$$\Lambda_p \xrightarrow{2} 2\Lambda_p \xrightarrow{2} 2^2 \Lambda_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m \Lambda_p \xrightarrow{2} \cdots \tag{4}$$

In our considerations we have direct systems of Minkowski balls, Minkowski domains and direct systems of critical lattices with respective maps and homomorphisms. Let $\mathbb{Q}_2$ and $\mathbb{Z}_2$ be respectively the field of 2-adic numbers and its ring of integers. Denote the corresponding direct limits by D_p^{dirlim} and by Λ_p^{dirlim}.

Proposition 6. $D_p^{dirlim} = \varinjlim 2^m D_p \in (\mathbb{Q}_2/\mathbb{Z}_2) D_p = (\bigcup_m \frac{1}{2^m} \mathbb{Z}_2/\mathbb{Z}_2) D_p$.

Proposition 7. $\Lambda_p^{dirlim} = \varinjlim 2^m \Lambda_p \in (\mathbb{Q}_2/\mathbb{Z}_2) \Lambda_p = (\bigcup_m \frac{1}{2^m} \mathbb{Z}_2/\mathbb{Z}_2) \Lambda_p$.

References

[1] H. Minkowski, *Diophantische Approximationen*, Leipzig: Teubner (1907).

[2] L.J. Mordell, Lattice points in the region $|Ax^4 + By^4| \leq 1$, *J, London Math. Soc.* **16** (1941), 152–156.

[3] C. Davis, Note on a conjecture by Minkowski, *J. London Math. Soc.*, **23**, 172–175 (1948).

[4] H. Cohn, Minkowski's conjectures on critical lattices in the metric $\{|\xi|^p + |\eta|^p\}^{1/p}$, *Annals of Math.*, **51**, (2), 734–738 (1950).

[5] G. Watson, Minkowski's conjecture on the critical lattices of the region $|x|^p + |y|^p \leq 1$, (I), (II), *Jour. London Math. Soc.*, **28**, (3, 4), 305–309, 402–410 (1953).

[6] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer, NY (1997).

[7] L.S. Pontryagin, *Select Works Volume 1*, CRC Press, Boca Raton London NY (2019).

[8] N. Glazunov, A. Golovanov, A. Malyshev, Proof of Minkowski's hypothesis about the critical determinant of $|x|^p + |y|^p < 1$ domain , Research in Number Theory 9. Notes of scientific seminars of LOMI. **151**(1986), Nauka, Leningrad, 40–53.

[9] N. Glazunov, On packing of Minkowski balls, Comptes rendus de l'Acad´emie bulgare Sci., Tome 76, No 3 (2023), 335-342.