On polynomial and regular maps of spheres

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This talk offers some results on to the intersection of algebraic topology and algebraic geometry.

Let K be a field and $X \subseteq K^m$, $Y \subseteq K^n$ algebraic sets. Recall that a map $f = (f_1, \ldots, f_n) : X \to Y$ is called *polynomial* (resp. *regular*) if there are polynomials $F_i, G_i \in \mathbb{R}[X_1, \ldots, X_m]$ such that $f_i(x) = F_i(x)$ (resp. $f_i(x) = \frac{F_i(x)}{G_i(x)}$, $G_i(X) \neq 0$) with $i = 1, \ldots, n$ for $x \in X$.

Remark 1. If K is a algebraically closed field then the only regular maps of algebraic sets are polynomial maps.

Example 2. (1) Let $K = \mathbb{R}$ or $\mathbb{C}$, the fields of reals or complex numbers. The n-sphere

$$\mathbb{S}^n(K) = \{(x_1, \ldots, x_{n_1}) \in \mathbb{K}^{n+1};\ x_1^2 + \cdots + x_{n+1}^2 = 1\} = V(X_0^2 + \cdots + X_n^2 - 1)$$

is an algebraic set in $\mathbb{K}^{n+1}$. Write $\mathbb{S}^n(\mathbb{R}) = \mathbb{S}^n$ and notice a diffeomorphism $\mathbb{S}^n(\mathbb{C}) \approx T\mathbb{S}^n$, the tangent bundle of $\mathbb{S}^n$. Consequently, a homotopy equivalence $\mathbb{S}^n(\mathbb{C}) \simeq \mathbb{S}^n$.

(2) Let $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ with the skew $\mathbb{R}$-algebra $\mathbb{H}$ of quaternions. The Grassmannian (of r-planes in K^n), can be identified with $G_{n,r}(K) = \{A \in M_n(K);\ A^2 = A,\ \bar{A} = A^t,\ \mathrm{rk}(A) = r\}$ for the set $M_n(K)$ of all $n \times n$-matrices over K.

But, for any idempotent $n \times n$ matrix over K, its rank coincides with the trace. Therefore, $G_{n,r}(K)$ can be viewed as a real affine variety.

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ be algebraic sets. Write $[X, Y]$ for the set of homotopy classes of continuous maps and $[X, Y]_{alg}$ the subset of $[X, Y]$ represented by regular maps. One of the main purposes of the talk is to estimate the size of $\pi_m(\mathbb{S}^n)_{alg} = [\mathbb{S}^m, \mathbb{S}^n]_{alg}$ in $\pi_m(\mathbb{S}^n) = [\mathbb{S}^m, \mathbb{S}^n]$.

Basing on [1], [3] and [4], we aim to show:

Theorem 3. *If $k = 0, 1, \ldots, 7$ then elements of $\pi_{n+k}(\mathbb{S}^n)$ can be represented by regular maps for $n \geq 1$.*

Next, we make use of [2] to show a homeomorphism $TG_{n,r}(K) \xrightarrow{\approx} \mathrm{Idem}_{n,r}(K)$ for the tangent bundle $TG_{n,r}(K)$ of $G_{n,r}(K)$ and $\mathrm{Idem}_{n,r}(K)$, the set of all idempotent $n \times n$ matrices with rank r for $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. Finally, we present:

Theorem 4. *If $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ then there is:*

(1) *a regular deformation retraction* $\mathrm{Idem}_{n,r}(K) \to G_{n,r}(K)$*;*

(2) *an injection* $\mathcal{P}_{\mathbb{C}}[V_{\mathbb{C}}, \mathrm{Idem}_{n,r}(K)] \to \mathcal{R}_{\mathbb{R}}[V, G_{n,r}(K)]$ *from the sets of homotopy classes of complex-valued polynomial to such a set of real-valued regular maps, where $V_{\mathbb{C}}$ denotes the Zariski closure in the affine space $\mathbb{C}^n$ of a subset $V \subseteq \mathbb{R}^n$.*

References

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