On some non-associative hyper-algebraic structures

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In this paper, new hyper-algebraic structures called hyperloop, multiloop, polyquasigroup and polyloop, and a special class of polyloop called right Bol polyloop are introduced and studied. It is shown that for any non-commutative (groupoid, quasigroup, loop), commutative and non-commutative (polygroupoid, polyquasigroup, polyloop) can be constructed. It is shown that a right Bol polyloop is characterized by any of seven equivalent identities and has the right alternative properties. Two examples of right Bol loops were constructed with the aid of a ring.

The newly introduced hyper-algebraic structures are:

Definition 1. (Polygroupoid, Polyquasigroup, Polyloop, Multiloop)

Let $\mathcal{M} = (P, \cdot)$ be a polygroupoid. Let $e \in P$ and $\diagup : P \times P \to \mathfrak{P}^*(H)$ and $\diagdown : P \times P \to \mathfrak{P}^*(H)$ such that

- **(a):** (i) $x \in (x \cdot y)\diagup y$ (ii) $x \in (x\diagup y) \cdot y$ (iii) $x \in y\diagdown(y \cdot x)$ (iv) $x \in y \cdot (y\diagdown x)$ for all $x, y \in P$, then $(P, \cdot, \diagdown, \diagup)$ will be called a polyquasigroup.
- **(b):** $x \cdot e = e \cdot x = x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a polyloop.
- **(c):** $x \in x \cdot e = e \cdot x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a multiloop.
- **(d):** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyloop. Then $(P, \cdot, \diagdown, \diagup)$ will be called an associative polyloop.

Definition 2. (Right Bol Polyloop)

Let $\mathcal{M} = (P, \cdot, \diagdown, \diagup, e)$ be a polyloop, then $(P, \cdot, \diagdown, \diagup, e)$ will be called a right Bol Polyloop, if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y) \ \forall \ x, y, z \in P \tag{1}$$

Result on equivalence between the hyper-algebraic structures in Definition 1 and some existing ones in literature is presented in Theorem 3.

Theorem 3. *Let $(G, \cdot)$ be a polygroupoid.*

1. *The following are equivalent:*
   - (a) *$(G, \cdot)$ is an hyperquasigroup.*
   - (b) *$(G, \cdot, \diagdown, \diagup)$ is a polyquasigroup.*
   - (c) *$(G, \cdot)$ is an quasigrouphypergroup.*
   - (d) *There exist hyperoperations $\diagdown$ and $\diagup$ on G such that $z \in x \cdot y \Leftrightarrow x \in z\diagup y \Leftrightarrow y \in x\diagdown z$ holds for all $x, y, z \in G$.*
2. *$(G, \cdot, e)$ is a hyperloop if and only if it $(G, \cdot, e)$ is a multiloop.*
3. *$(G, \cdot)$ is a hypergroup if and only if it is an associative polyquasigroup.*

(4) $(G,\cdot)$ *is an* H_v*-group if and only if it is a polyquasigroup with WASS.*
(5) $(G,\cdot)$ *is a Marty-Moufang hypergroup (*H_m*-group) if and only if it is a Moufang polyquasigroup. (Marty-Moufang hypergroup of Bayon and Lygeros* [1]*)*
(6) $(G,\cdot)$ *is a polygroup if and only if it is a associative polyloop.*

Theorem 4 describes a method of construction of commutative and non-commutative polyquasigroups (polyloops) using a non-commutative quasigroup (loop).

Theorem 4. *(Construction of polygroupoid, polyquasigroup and polyloop)*
Given a non-commutative groupoid (quasigroup, loop) $(G,\cdot,\diagdown,\diagup,e)$*, define an hyperoperation* $\odot : G\times G\to\mathfrak{P}^*(G)$ *as* $x\odot y=\{xy,yx\}$*. Then, there exist left division and right division hyperoperations* $\leftthreetimes : G\times G\to\mathfrak{P}^*(G)$ *and* $\rightthreetimes : G\times G\to\mathfrak{P}^*(G)$ *of* $\odot$ *such that* $x\leftthreetimes y=\{x\diagdown y, y\diagup x\}$ *and* $x\rightthreetimes y=\{x\diagup y, y\diagdown x\}$ *respectively and*
(1) $(G,\odot)$ *is a commutative polygroupoid.*
(2) $(G,\odot,\leftthreetimes,\rightthreetimes)$ *is a commutative polyquasigroup while* $(G,\leftthreetimes,\odot,\leftthreetimes)$ *and* $(G,\rightthreetimes,\rightthreetimes,\odot)$ *are non-commutative polyquasigroups.*
(3) $(G,\odot,\leftthreetimes,\rightthreetimes,e)$ *is a commutative polyloop while* $(G,\leftthreetimes,\odot,\leftthreetimes)$ *and* $(G,\rightthreetimes,\rightthreetimes,\odot)$ *are non-commutative polyquasigroups.*

Theorem 5 presents some results on the algebraic properties and characterization of right Bol polyloop as defined by (1) of Definition 2.

Theorem 5. *Let* $(P,\cdot,\diagdown,\diagup,e)$ *be a polyloop.* $(P,\cdot,\diagdown,\diagup,e)$ *is a right Bol polyloop if and only if any of the following is true (i)* $X(yz\cdot y)=(Xy\cdot z)y$ *(ii)* $x(yZ\cdot y)=(xy\cdot Z)y$ *(iii)* $x(Yz\cdot Y)=(xY\cdot z)Y$ *(iv)* $X(yZ\cdot y)=(Xy\cdot Z)y$ *(v)* $X(Yz\cdot Y)=(XY\cdot z)Y$ *(vi)* $x(YZ\cdot Y)=(xY\cdot Z)Y$ *(vii)* $X(YZ\cdot Y)=(XY\cdot Z)Y$ *for all* $x,y,z\in P$ *and* $X,Y,Z\subseteq P$*.*

Example 6. Let $(\mathbb{Z}_2,+,\cdot)$ be the ring of integer modulo 2 and let $G=\mathbb{Z}_2^3$. For (i,j,k) and (p,q,r) in G, define

$$(i,j,k)*(p,q,r)=(i+p,j+q,k+r+jpq).$$

Consider $\mathbb{Z}_2^3//N\subseteq P(\mathbb{Z}_2^3)$ where $N=N(\mathbb{Z}_2^3,*)=\{(0,0,0),(0,1,0),(1,0,0),(0,1,1)\}$ is the nucleus of $(\mathbb{Z}_2^3,*)$ so that

$$\mathbb{Z}_2^3//N=\Big\{\big\{(i,j,k),(i,j+1,k),(i,j,k+1),(i+1,j,k),(i,j+1,k+1)\big\}\ \mid\ i,j,k\in\mathbb{Z}_2\Big\}.$$

Define an hyperoperation $'\circ'$ on $\mathbb{Z}_2^3//N$ as follows

$$(i,j,k)N\circ(p,q,r)N=\Big\{\Big\{\big(i+a+p,j+b+q,k+c+jab+r+(j+b)pq\big),$$
$$\big(i+a+p,j+b+q+1,k+c+jab+r+(j+b)pq\big),\big(i+a+p,j+b+q,k+c+jab+r+$$
$$(j+b)pq+1\big),\big(i+a+p+1,j+b+q,k+c+jab+k+(j+b)pq\big),$$
$$\big(i+a+p,j+b+q+1,k+c+jab+r+(j+b)pq+1\big)\Big\}\ \Big|\ i,j,k,p,q,r\in\mathbb{Z}_2,a,b,c\in N\Big\}.$$

Then, $\Big(\mathbb{Z}_2^3//N,\circ\Big)$ is a right Bol polyloop.

References

[1] Bayon R. and Lygeros N. The H_v-groups and Marty-Moufang hypergroups. *Proceedings of the 1st International Conference on Algebraic Informatics*, Aristotle Univ. Thessaloniki, Thessaloniki, 285–294, 2005.
[2] Oyebola, O.O. and Jaiyéọlá, T. G. Non-associative algebraic hyperstructures and their applications to biological inheritance. *Monografías Matemáticas García de Galdeano* 42: 229–-241, 2019.