In this paper, new hyper-algebraic structures called hyperloop, multiloop, polyquasigroup and polyloop, and a special class of polyloop called right Bol polyloop are introduced and studied. It is shown that for any non-commutative (groupoid, quasigroup, loop), commutative and non-commutative (polygroupoid, polyquasigroup, polyloop) can be constructed. It is shown that a right Bol polyloop is characterized by any of seven equivalent identities and has the right alternative properties. Two examples of right Bol loops were constructed with the aid of a ring.

The newly introduced hyper-algebraic structures are:

Definition 1. (Polygroupoid, Polyquasigroup, Polyloop, Multiloop)

Let \( M = (P, \cdot) \) be a polygroupoid. Let \( e \in P \) and \( \lnot : P \times P \to P^* \) \( \setminus : P \times P \to P^* \) such that

(a): (i) \( x \in (x \cdot y) \setminus y \) (ii) \( x \in (x/ y) \cdot y \) (iii) \( x \in y \setminus (y \cdot x) \) (iv) \( x \in y \cdot (y \setminus x) \) for all \( x, y \in P \), then \( (P, \cdot, \lnot, \setminus) \) will be called a polyquasigroup.

(b): \( x \cdot e = e \cdot x = x \) for all \( x \in P \) and \( (P, \cdot, \lnot, \setminus) \) is a polyquasigroup. Then \( (P, \cdot, \lnot, \setminus, e) \) will be called a polyloop.

(c): \( x \in x \cdot e = e \cdot x \) for all \( x \in P \) and \( (P, \cdot, \lnot, \setminus) \) is a polyquasigroup. Then \( (P, \cdot, \lnot, \setminus, e) \) will be called a multiloop.

(d): \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) for all \( x, y, z \in P \) and \( (P, \cdot, \lnot, \setminus) \) is a polyloop. Then \( (P, \cdot, \lnot, \setminus) \) will be called an associative polyloop.

Definition 2. (Right Bol Polyloop)

Let \( M = (P, \cdot \lnot, \setminus, e) \) be a polyloop, then \( (P, \cdot \lnot, \setminus, e) \) will be called a right Bol Polyloop, if it satisfies the identity

\[
(xy \cdot z)y = x(yz \cdot y) \forall x, y, z \in P
\]

Result on equivalence between the hyper-algebraic structures in Definition 1 and some existing ones in literature is presented in Theorem 3. and some existing ones in literature is presented in Theorem 3.

Theorem 3. Let \( (G, \cdot) \) be a polygroupoid.

(1) The following are equivalent:

(a) \( (G, \cdot) \) is an hyperquasigroup.

(b) \( (G, \cdot, \lnot, \setminus) \) is a polyquasigroup.

(c) \( (G, \cdot) \) is a quasigroup hypergroup.

(d) There exist hyperoperations \( \lnot \) and \( \setminus \) on \( G \) such that \( z \in x \cdot y \iff x \in z/y \iff y \in x \setminus z \) holds for all \( x, y, z \in G \).

(2) \( (G, \cdot, e) \) is a hyperloop if and only if it \( (G, \cdot, e) \) is a multiloop.

(3) \( (G, \cdot) \) is a hypergroup if and only if it is an associative polyquasigroup.
(4) \((G, \cdot)\) is an \(H_u\)-group if and only if it is a polyquasigroup with WASS.

(5) \((G, \cdot)\) is a Marty-Moufang hypergroup (\(H_m\)-group) if and only if it is a Moufang polyquasigroup. (Marty-Moufang hypergroup of Bayon and Lygeros [1])

(6) \((G, \cdot)\) is a polygroup if and only if it is a associative polyloop.

Theorem 4 describes a method of construction of commutative and non-commutative polyquasigroups (polyloops) using a non-commutative quasigroup (loop).

Theorem 4. (Construction of polygroupoid, polyquasigroup and polyloop)

Given a non-commutative groupoid (quasigroup, loop) \((G, \cdot, \backslash, /, e)\), define an hyperoperation \(\circ : G \times G \to \mathcal{P}^*(G)\) as \(x \circ y = \{xy, yx\}\). Then, there exist left division and right division hyperoperations \(\land : G \times G \to \mathcal{P}^*(G)\) and \(\land : G \times G \to \mathcal{P}^*(G)\) of \(\circ\) such that \(x \land y = \{x \backslash y, y \backslash x\}\) and \(x \land y = \{x \backslash y, y \backslash x\}\) respectively and

1. \((G, \circ)\) is a commutative polygroupoid.
2. \((G, \circ, \land, \land)\) is a commutative polyquasigroup while \((G, \land, \circ, \circ)\) are non-commutative polyquasigroups.
3. \((G, \circ, \land, \land, e)\) is a commutative polyloop while \((G, \land, \circ, \circ)\) and \((G, \land, \circ, \land)\) are non-commutative polyloops.

Theorem 5 presents some results on the algebraic properties and characterization of right Bol polyloop as defined by [1] of Definition 2.

Theorem 5. Let \((P, \cdot, \backslash, /, e)\) be a polyloop. \((P, \cdot, \backslash, /, e)\) is a right Bol polyloop if and only if any of the following is true (i) \(X(yz \cdot y) = (Xy \cdot z)y\) (ii) \(x(yZ \cdot y) = (xy \cdot Z)Y = (xY \cdot z)Y\) (iv) \(X(yZ \cdot y) = (Xy \cdot Z)y\) (v) \(X(Yz \cdot Y) = (XY \cdot Z)Y\) (vi) \(x(Yz \cdot Y) = (xY \cdot Z)Y\) (vii) \(X(Yz \cdot Y) = (XY \cdot Z)Y\) for all \(x, y, z \in P\) and \(X, Y, Z \subseteq P\).

Example 6. Let \((\mathbb{Z}_2, +, \cdot)\) be the ring of integer modulo 2 and let \(G = \mathbb{Z}_2^3\). For \((i, j, k)\) and \((p, q, r)\) in \(G\), define

\[(i, j, k) \ast (p, q, r) = (i + p, j + q, k + r + jpq).\]

Consider \(\mathbb{Z}_2^3 / N \subseteq P(\mathbb{Z}_2^3)\) where \(N = N(\mathbb{Z}_2^3, \ast) = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1)\}\) is the nucleus of \((\mathbb{Z}_2^3, \ast)\) so that

\[\mathbb{Z}_2^3 / N = \{(i, j, k), (i, j, k + 1), (i, j, k + 1), (i + 1, j, k), (i, j + 1, k + 1)\} \mid i, j, k \in \mathbb{Z}_2\}.

Define an hyperoperation \(\circ\) on \(\mathbb{Z}_2^3 / N\) as follows

\[(i, j, k)N \circ (p, q, r)N = \{(i + a + p, j + b + q, k + c + jab + r + (j + b)pq),
(i + a + p, j + b + q + 1, k + c + jab + r + (j + b)pq),
(i + a + p, j + b + q, k + c + jab + r + (j + b)pq + 1),
(i + a + p + 1, j + b + q, k + c + jab + k + (j + b)pq),
(i + a + p, j + b + q + 1, k + c + jab + r + (j + b)pq + 1)\} \mid i, j, k, p, q, r \in \mathbb{Z}_2, a, b, c \in N\}.

Then, \(\mathbb{Z}_2^3 / N, \circ\) is a right Bol polyloop.

REFERENCES
