Explicit construction of explicit real algebraic functions and real algebraic manifolds via Reeb graphs

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In this talk, we present explicit real algebraic functions on explicit real manifolds via *Reeb graphs*. Our study is mainly motivated by real algebraic geometry, pioneered by Nash and Tognoli for example. A smooth closed manifold can be regarded as a non-singular real algebraic manifold. Existence theory of real algebraic manifolds and real algebraic maps has been also well-known. We can easily attain such objects and morphisms whereas it is very difficult to know explicitly. For more precise exposition, see [4] for example.

Of course some specific exmaples of real algebraic maps are well-known. Canonical embeddings and projections of unit spheres are simplest examples. As functions which are in considerable cases regarded as generalized ones, Lie groups and so-called *symmetric spaces* have nice functions represented as real polynomial functions. See [7] for example. However, it is difficult to know their global structures and properties explicitly in general.

Problem 1. Can we know global structures and properties of the functions and maps. For example, can we know information on preimages?

For this, we consider the following problem, established in [10]. This comes from singularity theory of smooth maps and applications to differential topology of manifolds. The *Reeb graph* of a smooth function is the graph whose underlying space is the natural quotient space of the manifold and consists of all connected components of preimages. Its vertex set is the set of all connected components containing some singular points of the function. As [9] shows, for smooth functions with finitely many singular values, we can have such graphs. [8] is a pioneering paper on this notion. Reeb graphs have some information of the manifolds nicely and fundamental and strong tools in geometry of manifolds.

Problem 2. Can we reconstruct a nice smooth function on some manifold whose Reeb graph is the given graph? We do not fix the manifold beforehand.

[10] constructs desired functions on closed surfaces for some nice graphs. [5] extends this to arbitrary finite graphs. [6] considers such a problem for a certain class of finite graphs and Morse functions such that connected components of preimages having no singular points are spheres. Our study [1] considers the following problem first. It is for functions on 3-dimensional closed manifolds. [9] presents a related general result through our informal discussions on [1].

Problem 3. Can we construct the function in Problem 2 with prescribed preimages?

This talk is on answers to the following problem, pioneered by the speaker first in [2].

Problem 4. Can we construct these functions and the manifolds in finer categories such as the real analytic category and the real algebraic category, for example?

We present our main results with several notions we need. An *algebraic domain* D is a bounded open set in the real affine space $\mathbb{R}^k$ surrounded by finitely many mutually disjoint non-singular connected real algebraic hypersurfaces each S_j of which is the zero set of some real polynomial f_j. The *Poincaré-Reeb graph* of it is a canonically obtained graph whose underlying space is the natural quotient space of the closure $\overline{D}$ of the domain and consists of all connected components of preimages for the restriction of the projection $\pi(x_1, \cdots, x_k) := x_1$ to $\overline{D}$. Its vertex set is defined as the set of all connected components containing some singular points of the function $\pi|_{\overline{D}-D}$.

Theorem 5 ([2]). *Let G be a Poincaré-Reeb graph of $D \subset \mathbb{R}^k$. Let D be an algebraic domain represented as the intersection $\bigcap\{x \in \mathbb{R}^k \mid f_j(x) > 0\}$. Then we can construct a smooth real algebraic function whose Reeb graph is isomorphic to G on some non-singular real algebraic closed manifold.*

Example 6. Any Poincaré-Reeb graph G of any bounded connected open set $D \subset \mathbb{R}^k$ surrounded by finitely many mutually disjoint spheres of fixed radii satisfies the assumption of Theorem 5. See also FIGURE 1 of [2].

In the proof, first we construct a nice smooth real algebraic map into $\mathbb{R}^k$ whose image is $\overline{D}$. More precisely, we construct one such that the preimage of a point in the boundary is a one-point set and that the preimage of a point in the interior is a sphere. Last we compose the projection.

Theorem 7 ([3]). *Let $l > 3$ and $m > 2$ be integers. Let $\{t_j\}_{j=1}^{l}$ be an increasing sequence of real numbers. Let $\{F_j\}_{j=1}^{l-1}$ be a family of smooth manifolds satisfying the following conditions.*

- *F_1 and F_{l-1} are diffeomorphic to the $(m-1)$-dimensional unit spheres S^{m-1}.*
- *The others are diffeomorphic to S^{m-1} or represented as connected sums of finitely many manifolds diffeomorphic to the products $S^j \times S^{m-j-1}$ for some integers $1 \leq j \leq m-2$: the connected sum is taken in the smooth category. For adjacent integers $1 \leq j \leq l-2$ and $j+1$, either F_j or F_{j+1} is not diffeomorphic to the unit sphere.*

Then we have an m-dimensional non-singular real algebraic closed and connected manifold M and a smooth real algebraic function $f : M \to \mathbb{R}$ such that the number of singular points is finite, that $\{t_j\}_{j=1}^{l}$ is the set of all singular values and that the preimage $f^{-1}(p_j)$ is diffeomorphic to F_j for $p_j \in (t_j, t_{j+1})$.

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