Conformal equivalence of 3-webs

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Let 3-web $W_3\left\langle\omega_1, w_2, \omega_3\right\rangle$ defined in a domain $D$ on the conformal plane $\left(\mathbb{R}^2, g\right)$. We say that this 3-web is *regular* in $D$ if in this domain:

(1) The discriminant
$$\widetilde{\Delta}=-4 I_2+I_1^2+18 I_1 I_2-4 I_1^3-27 I_2^3$$
differs from zero.

(2) Invariants
$$I_1=\frac{J_2}{J_1^2} \quad and \quad I_2=\frac{J_3}{J_1^3}$$
are functionally independent in the domain, that is, the differential 2-form $\Omega=d I_1 \wedge d I_2 \neq 0$. Moreover, invariants $I_1, I_2$ are coordinates in the domain.

We remark that the elementary symmetric functions
$$\begin{aligned}
J_1 & =\lambda_1+\lambda_2+\lambda_3, \\
J_2 & =\lambda_1 \lambda_2+\lambda_1 \lambda_3+\lambda_2 \lambda_3, \\
J_3 & =\lambda_1 \lambda_2 \lambda_3
\end{aligned}$$
are $S_3$ - invariants and $\lambda_1, \lambda_2, \lambda_3$ are positive smooth functions.

Let's number now forms $\omega_1, w_2, \omega_3$ in the domain and say that the 3-web is *oriented* in the domain if in this numbering $\omega_1 \wedge \omega_2=r_{12} \Omega$, where $r_{12}>0$. In this case we'll scale forms $\omega_i$ in such a way, that
$$\omega_1 \wedge \omega_2=\Omega .$$
In opposite case, we call the 3-web *non-oriented* and scale the 1-forms $\omega_i$ in such a way, that
$$\omega_1 \wedge \omega_2=-\Omega .$$

In these both cases we decompose 1-forms $\omega_i$ in the invariant coordinates $I_1, I_2$
$$\omega_i=\sum_{j=1}^2 w_{i j} d I_j,$$
Then, all functions $w_{i j}, i=1,2,3 ; j=1,2$ are conformal invariants, satisfying the following additional relations
$$\sum_{i=1}^3 w_{i j}=0,\ \ j=1,2 .$$

Now, let's write down the standard metric tensor $g$ in invariant coordinates as follows
$$g=\sum_{i, j=1}^2 g_{i j} d I_i \otimes d I_j .$$
Remark, that the volume 2-form $\Omega_g$, associated with metric $g$, is the following
$$\Omega_g=\sqrt{\operatorname{det}\left\|g_{i j}\right\|} d I_1 \wedge d I_2 .$$

Therefore, the metric tensor

$$\widetilde{g} = \frac{g}{\sqrt{\det \|g_{ij}\|}}$$

has the associated volume form $\Omega_{\widetilde{g}} = \Omega$.

Finally, we get the following result.

**Theorem 1.** *Let 3-web $W_3 \langle \omega_1, w_2, \omega_3 \rangle$ be regular in a domain $D$ in the conformal plane $\left(\mathbb{R}^2, g\right)$. Then the above functions*

$$w_{ij}, \qquad \widetilde{g}_{ij} = \frac{g_{ij}}{\sqrt{\det \|g_{ij}\|}}$$

*that are components of 1-forms $\omega_1, \omega_2$ and the metric tensor $\widetilde{g}$ in the invariant coordinates $I_1, I_2$, are conformal invariants of plane 3-webs.*

*Moreover, any two regular 3-webs are conformly equivalent if and only if the corresponding functions $w_{ij}$ and $\widetilde{g}_{ij}$ coinside.*