Deformational symmetries of functions with isolated singularities on the Mobius band

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Let M be a smooth compact 2-dimensional manifold which have a non-empty boundary, and P be either a real line or a circle. Denote by $D(M,Y)$ the group of diffeomorphisms of M fixed on a closed subset $Y \subset M$. There is a natural right action of the group $D(M,Y)$ on the space of smooth functions $C^{\infty}(M,\mathbb{R})$ defined by the following rule: $(h,f) \longmapsto f \circ h$, where $h \in D(M,Y)$, $f \in C^{\infty}(M,\mathbb{R})$.

Let

$$\mathcal{O}(f,Y) = \{f \circ h \mid h \in \mathcal{D}(M,Y)\}$$

be the *orbit* of f under this action. Endow $C^{\infty}(M,\mathbb{R})$ with Whitney C^{∞}-topology and $\mathcal{O}(f,Y)$ with induced one.

Definition 1. Denote by $\mathcal{F}(M,P)$ the space of smooth maps $f \in C^{\infty}(M,P)$ having the following properties:

(1) the map f takes constant values at each connected component of ∂M and has no critical points on it;
(2) for every critical point z of f there is a local presentation $f_z \colon \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Definition 2. Let G, H be groups, $m \in \mathbb{Z}$ and $\gamma \colon H \to H$ be automorphism of order 2. Define the automorphism $\phi \colon G^{2m} \times H^m \to G^{2m} \times H^m$ by the formula

$$\phi(g_0, \dots, g_{2m-1}, h_0, \dots, h_{m-1}) = (g_{2m-1}, g_0, \dots, g_{2m-2}, h_1, h_2, \dots, h_{m-1}, \gamma(h_0)).$$

This automorphism ϕ generates homomorphism $\phi' \colon \mathbb{Z} \to G^{2m} \times H^m$. The corresponding semidirect product $G^{2m} \times H^m \rtimes_{\phi'} \mathbb{Z}$ will be denoted $(G,H) \wr_{\gamma,m} \mathbb{Z}$.

Definition 3. Let $\mathcal{P}$ be a minimal class of groups satisfying the following conditions:

1) $1 \in \mathcal{P}$;
2) if $A, B \in \mathcal{P}$, then $A \times B \in \mathcal{P}$;
3) if $A \in \mathcal{P}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{P}$.

It was shown in [2] that if M has negative Euler characterictic, then fundamental groups of orbits of functions in $\mathcal{F}(M,P)$ are direct products of such groups for functions only on cylinders, disks and Möbius bands. Moreover, if M is either a 2-disk or a cylinder, then $\pi_1\mathcal{O}(f,\partial M) \in \mathcal{P}$.

Theorem 4. *Let M be a Möbius band and let $f \in \mathcal{F}(M,P)$. Then*

$$\pi_1\mathcal{O}(f,\partial M) \cong A \times (G,H) \wr_{\gamma,m} \mathbb{Z}, \text{ where } A, G, H \in \mathcal{P}.$$

References

[1] Iryna Kuznietsova, Sergiy Maksymenko. Homotopy properties of smooth functions on the Möbius band, *Proceedings of the International Geometry Center*, vol. 12, no. 3, 2019.

[2] Maksymenko S. I. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. *Topology and its Applications*, vol. 282, 2020.