# Twisted Sasaki metric on the unit tangent bundle and harmonicity

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Let $(M^n, g)$ be an n-dimensional Riemannian manifold, $TM^n$ be its tangent bundle, $\mathfrak{X}(M^n)$ be the Lie algebra of smooth vector fields of a Riemannian manifold $(M^n, g)$, $\nabla$ be the Levi-Civita connection on $M^n$. The standard metric on the tangent bundle of Riemannian manifold $(M^n, g)$ is the Sasaki metric [7]. It can be completely defined by scalar products of various combinations of vertical and horizontal lifts of vector fields. The Sasaki metric weakly inherits the base manifold properties. That is why the rigidity of Sasaki metric motivates many authors consider various deformations of Sasaki metric (see [1], [3] and others).

Belarbi L. and El Hendi H. introduce in [2] the twisted Sasaki metric on the tangent bundle $TM$ as a new natural metric non-rigid on $TM$. The twisted Sasaki metric is defined as follows.

**Definition 1.** [2] Let $(M^n, g)$ be a Riemannian manifold and $\delta, \varepsilon : M^n \longrightarrow \mathbb{R}$ be strictly positive smooth functions. On the tangent bundle $TM^n$, we define a twisted Sasaki metric noted $G^{\delta,\varepsilon}$ by

$$\begin{aligned}
&G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^h) = \delta(x) g_x(X, Y),\\
&G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^v) = 0,\\
&G^{\delta,\varepsilon}_{(x,\xi)}(X^v, Y^v) = \varepsilon(x) g_x(X, Y).
\end{aligned}$$

for all vector fields $X, Y \in \mathfrak{X}(M^n)$ and $(x, \xi) \in TM^n$.

Note that, if $\delta = \varepsilon = 1$, then $G^{\delta,\varepsilon}$ is the Sasaki metric [7]. If $\delta = 1$, then $G^{\delta,\varepsilon}$ is the vertical rescaled metric (see [3], [4]).

For a unit vector field $\xi$ on a compact Riemannian manifold $(M, g)$, Gerrit Weigmink [8] considered a very natural geometric functional

$$\int_M ||A_\xi||^2 dVol(M),$$

where $||A_\xi||$ is a norm of the Nomizu operator $A_\xi X = -\nabla_X \xi$, i.e. $||A_\xi||^2 = \sum_{i=1}^n g(A_\xi e_i, A_\xi e_i)$ relative to some orthonormal frame $(e_1, \dots, e_n)$. It was proved, that this functional is unbounded from above. The critical points of this functional was called *harmonic unit vector fields*. G. Wigmink proved, that a unit vector field $\xi$ on compact Riemannian manifold is harmonic if and only if

$$\bar{\Delta}\xi = ||A_\xi||^2 \xi,$$

where $\bar{\Delta}\xi$ is rough Laplacian (or Bochner Laplacian) of the field $\xi$ defined as $\bar{\Delta}\xi = -trace \nabla^2 \xi$, where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

On the other hand (see [5]), the energy of a *mapping* $\phi : (M^n, g) \to (N^k, h)$ between Riemannian manifolds is defined as

$$E(\phi) := \frac{1}{2} \int_M |d\phi|^2 dVol_M.$$

The mapping $\phi$ *is called harmonic* if it the critical point of the energy functional. It was proved that the mapping $\phi$ is harmonic if and only if the divergence of its differential vanishes, or equivalently its tension field $\tau(\phi) = div(d\phi)$ vanishes identically, where $|d\phi|$ is a norm of 1-form $d\phi$ in the cotangent bundle $T^*M^n$. Supposing on $T_1 M^n$ the Sasaki metric $g_S$, a unit vector field $\xi$ as a mapping $\xi : (M^n, g) \to (T_1 M^n, g_S)$ *defines a harmonic map* if and only if it is *harmonic* and, in addition, $\sum_{i=1}^n R(\xi, A_\xi e_i) e_i = 0$ relative to some othonormal frame $\{e_i\}_{i=1}^n$.

In the present research we define the twisted Sasaki metric [2] on the unit tangent bundle $T_1M^n$ of n-dimensional Riemannian manifold $(M^n, g)$, obtain Kowalski-type formulas, calculate the tension field of the mapping $\xi\colon\ M^n \to (T_1M^n, G^{\delta,\varepsilon})$. As a main result, for twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle $T_1M^n$ of n-dimensional Riemannian manifold $(M^n, g)$ we obtain the necessary and sufficient conditions for harmonicity of left-invariant unit vector field $\xi$ and mapping $\xi\colon\ M^n \to (T_1M^n, G^{\delta,\varepsilon})$.

**Theorem 2.** *Unit vector field $\xi$ is harmonic on n-dimensional Riemannian manifold $(M^n, g)$ equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle $T_1M^n$ if and only if*

$$\bar{\Delta}\xi + \frac{1}{\varepsilon}A_\xi(\nabla\varepsilon) = ||A_\xi||^2\xi. \tag{1}$$

*Harmonic unit vector field $\xi$ defines a harmonic mapping $\xi\colon\ M^n \to (T_1M^n, G^{\delta,\varepsilon})$ on n-dimensional Riemannian manifold $(M^n, g)$ equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle $T_1M^n$ if and only if*

$$2\varepsilon \cdot trace(Hm_\xi) + (n-2)\nabla\delta + ||A_\xi||^2\nabla\varepsilon = 0. \tag{2}$$

As an examples, we consider the necessary and sufficient conditions for harmonicity of left-invariant unit vector field $\xi$ and harmonic mapping $\xi\colon\ M^3 \to (T_1M^3, G^{\delta,\varepsilon})$ on 3-dimensional unimodular Lie group equipped with twisted Sasaki metric on the unit tangent bundle $T_1M^3$, using orthonormal frame of Milnor J. [6]. In addition, we consider some examples of deformations that preserves existence harmonic left-invariant unit vector fields $\xi$ and harmonic mapping $\xi\colon\ M^3 \to (T_1M^3, G^{\delta,\varepsilon})$ on 3-dimensional unimodular Lie groups with the left invariant metric.

## References


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