# Lie structures of the Sheffer group over a Hilbert space

**Eugene Lytvynov**
(Swansea University, Bay Campus, Swansea, SA1 8EN, UK)
*E-mail:* e.lytvynov@swansea.ac.uk

Umbral calculus (also called calculus of finite differences) is essentially the theory of Sheffer polynomial sequences, which are characterised by the exponential form of their generating function. The class of Sheffer sequences includes the binomial sequences and Appell sequences. After a long period when one-dimensional umbral calculus was used for purely formal calculations, the theory became rigorous in the 1970s due to the seminal works of G.-C. Rota, S. Roman and their co-authors. Their theory is nowadays called the modern umbral calculus, see e.g. the monographs [4, 8]. Umbral calculus found applications in combinatorics, theory of special functions, approximation theory, probability and statistics, topology and physics, see e.g. the survey paper [2] for a long list of references. A central object of studies of umbral calculus is the umbral composition, which equips the set of all Sheffer sequences with a group structure. This group is isomorphic to the Riordan group of infinite lower triangular matrices [6, 10]. Recently, Cheon et al. [3] (see also Bacher [1]) introduced a Lie group structure on the Riordan group and found the corresponding Lie algebra.

A lot of research has been done to extend the classical umbral calculus to the multivariate case, see Section 4 in [2] for a list of references. However, this research had a significant drawback of being basis-dependent. The paper [5] developed foundations of infinite-dimensional, basis-independent umbral calculus.

In this talk, we will discuss Lie structures of the group of Sheffer polynomials over a Hilbert space. Let

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_0$$

be standard triple of real separable Hilbert spaces, i.e., the Hilbert space $\mathcal{H}_+$ is densely and continuously embedded into $\mathcal{H}_0$ and $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$, while the dual paring between elements of $\mathcal{H}_-$ and $\mathcal{H}_+$ is determined by the inner product in $\mathcal{H}_0$. Then, for each $n$, we also get a standard triple

$$\mathcal{H}_+^{\odot n} \subset \mathcal{H}_0^{\odot n} \subset \mathcal{H}_0^{\odot n}.$$

Here $\odot$ denotes the symmetric tensor product. For $F^{(n)} \in \mathcal{H}_-^{\odot n}$ and $f^{(n)} \in \mathcal{H}_+^{\odot n}$, we denote by $\langle F^{(n)}, f^{(n)}\rangle$ the dual pairing between $F^{(n)}$ and $f^{(n)}$. (For a real Hilbert space $\mathcal{H}$, we define $\mathcal{H}^{\odot 0} := \mathbb{R}$.)

A (continuous) polynomial on $\mathcal{H}_-$ is a function $p : \mathcal{H}_- \to \mathbb{R}$ of the form

$$p(\omega) = \sum_{i=0}^{n} \langle \omega^{\odot i}, f^{(i)}\rangle, \quad \omega \in \mathcal{H}_-,\ f^{(i)} \in \mathcal{H}_+^{\odot i}, \quad i = 0, 1, \dots, n,\ n \in \mathbb{N}_0. \tag{1}$$

We denote by $\mathcal{P}(\mathcal{H}_-)$ the vector space of all polynomials on $\mathcal{H}_-$. By identifying the polynomial $p(\omega)$ in (1) with the sequence $(f^{(i)})$, we endow $\mathcal{P}(\mathcal{H}_-)$ with the topology of the topological direct sum of the Hilbert spaces $\mathcal{H}_+^{\odot i}$, $i \in \mathbb{N}_0$.

A monic polynomial sequence on $\mathcal{H}_-$ is a continuous linear map $P \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-))$ that satisfies

$$(P\langle \cdot^{\odot n}, f^{(n)}\rangle)(\omega) = \sum_{i=0}^{n} \langle \omega^{\odot i}, p_{in} f^{(n)}\rangle, \tag{2}$$

where $p_{in} \in \mathcal{L}(\mathcal{H}_+^{\odot n}, \mathcal{H}_+^{\odot i})$ and $p_{nn} = \mathbf{1}$. Denote by $p_{in}^* \in \mathcal{L}(\mathcal{H}_+^{\odot i}, \mathcal{H}_+^{\odot n})$ the adjoint (dual) operator of $p_{in}$. Then

$$(P\langle \cdot^{\odot n}, f^{(n)}\rangle)(\omega) = \langle p^{(n)}(\omega), f^{(n)}\rangle,$$

where $p^{(n)}(\omega) \in \mathcal{H}_-^{\odot n}$ is given by $p^{(n)}(\omega) := \sum_{i=0}^n p_{in}^* \omega^{\odot i}$. Thus, $p^{(n)} : \mathcal{H}_- \to \mathcal{H}_-^{\odot n}$, and we can identify the linear operator $P$ from (2) with the sequence $(p^{(n)})_{n=0}^\infty$.

A monic polynomial sequence $(p^{(n)})_{n=0}^\infty$ is called a Sheffer sequence (on $\mathcal{H}_-$) if it has the exponential generating function of the form

$$\sum_{n=0}^\infty \frac{1}{n!} \langle p^{(n)}(\omega), \xi^{\odot n} \rangle = \exp\left[\langle \omega, B(\xi) \rangle\right] A(\xi), \quad \omega \in \mathcal{H}_-,\ \xi \in \mathcal{H}_+, \tag{3}$$

where $B(\xi) = \xi + \sum_{k=2}^\infty b_k \xi^{\odot k}$, $b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+)$, $A(\xi) = 1 + \sum_{k=1}^\infty a_k \xi^{\odot k}$, $a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R})$, and the equality (3) is understood as the equality of formal tensor power series in $\xi$, see [5]. We denote by $\mathbb{S}$ the set of all Sheffer sequences on $\mathcal{H}_-$. We also denote by $\mathbb{A}$ the set of all Appell sequences, i.e., the Sheffer sequences for which $B(\xi) = \xi$ in (3), and we denote by $\mathbb{B}$ the set of all the binomial sequences, i.e., the Sheffer sequences for which $A(\xi) = 1$ in (3).

Since elements of $\mathbb{S}$ were defined through continuous linear operators in $\mathcal{P}(\mathcal{H}_-)$, one can ask a natural question whether a product of two such operators yields a Sheffer sequence. The answer to this question is positive, and furthermore the set $\mathbb{S}$, equipped with this product, becomes a group. Note that the neutral element in this group is the identity operator, equivalently the monomial sequence $p^{(n)}(\omega) = \omega^{\odot n}$. Furthermore, both $\mathbb{A}$ and $\mathbb{B}$ are subgroups $\mathbb{S}$, $\mathbb{A}$ is a normal subgroup of $\mathbb{S}$, and the Sheffer group $\mathbb{S}$ is a semidirect product of the Appell group $\mathbb{A}$ and the binomial group $\mathbb{B}$.

In the talk, we will discuss the following results:

- We will show that $\mathbb{S}$, $\mathbb{A}$ and $\mathbb{B}$ can be described as infinite-dimensional Lie groups, in the sense of Milnor [7], see also [9, Chapter 3].
- We will find the explicit form of the Lie algebra of each of these Lie groups, and we will find a Lie bracket on them.
- We will conclude that the Sheffer group is constructed from two basic operations: gradient of polynomials on $\mathcal{H}_-$ and multiplication by $\omega$.

This is joint result with Dmitri Finkelshtein (Swansea University) and Maria João Oliveira (Universidade Aberta, Lisbon).

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