SKT Hyperbolic and Gauduchon Hyperbolic Compact Complex Manifolds

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Definition 1. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and ω be a metric on X: be a C^∞ positive definite $(1,1)$-form on X .

i) ω is *Kähler*, if $d\omega = 0$.
ii) ω is *balanced*, if $d\omega^{n-1} = 0$.
iii) ω is *Gauduchon*, if $\bar\partial\partial\omega^{n-1} = 0$, such a metric always exists on a compact complex manifold.
iv) ω is **SKT** (or pluriclosed), if $\partial\bar\partial\omega = 0$

Let $\pi_X : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} = \pi_X^\star\omega$ be the Hermitian metric on $\widetilde{X}$ that is the lift of ω. Recall that a C^∞ k-form α on X is said to be $\widetilde{d}$(bounded) with respect to ω if $\pi_X^\star\alpha = d\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-form β on $\widetilde{X}$ that is bounded w.r.t. $\widetilde{\omega}$. (See [1] and [2]). In general, we propose the following definition which generalizes that of $\widetilde{d}$-bounded of a differential form.

Definition 2. a C^∞ k-form ϕ on X is said to be $\widetilde{(\partial+\bar\partial)}$-bounded with respect to ω if $\pi_X^\star\phi = \partial\alpha + \bar\partial\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-forms α and β on $\widetilde{X}$ that are bounded w.r.t. $\widetilde{\omega}$.

M. Gromov introduced in one of his seminal papers [1] the notion of *Kähler hyperbolicty* for a compact Kähler manifold X. The manifold X is called *Kähler hyperbolic* if X admits a Kähler metric ω whose lift $\widetilde{\omega}$ to the universal cover $\tilde{X}$ of X can be expressed as

$$\widetilde{\omega} = d\alpha$$

for a *bounded* 1-form α on $\tilde{X}$. As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.
The Kähler hyperbolicity is generalized in [2] to what we call **balanced hyperbolicity**. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a *balanced metric*. Meanwhile, a compact complex n-dimensional manifold X is said to be balanced hyperbolic if it carries a balanced metric ω such that ω^{n-1} is $\widetilde{d}$-bounded. The Brody hyperbolicity is replaced by what we call **divisorial hyperbolicity**. A compact complex manifold X is called *divisorially hyperbolic* if there exists no non-trivial holomorphic map from $\mathbb{C}^{n-1}$ to X satisfying certain *subexponential volume growth condition*.

We introduce the following

Definition 3. Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. A Hermitian metric ω on X is said to be

(1) **SKT hyperbolic** if ω is SKT and $\widetilde{(\partial+\bar\partial)}$ − bounded with respect to ω. The manifold X is said to be SKT hyperbolic if it carries a *SKT hyperbolic metric*.
(2) **Gauduchon hyperbolic** if ω^{n-1} is $\widetilde{(\partial+\bar\partial)}$ − bounded with respect to ω. The manifold X is said to be Gauduchon hyperbolic if it carries a *Gauduchon hyperbolic metric*.

Lemma 4. *The following implication holds:*

X *is* **Kähler hyperbolic** $\Longrightarrow$ X *is* **SKT hyperbolic**

$\Downarrow$

X *is* **balanced hyperbolic** $\Longrightarrow$ X *is* **Gauduchon hyperbolic**

The following results are taken from [3]

Theorem 5. *Every* **SKT hyperbolic** *compact complex manifold is* **Kobayashi hyperbolic**.

Remark 6. An immediate observation is that, since a *SKT hyperbolic* manifold X contains no rational curves, then by Mori's cone theorem we get K_X is nef.

Theorem 7. *Every* **Gauduchon hyperbolic** *compact complex manifold is* **divisorially hyperbolic**.

Theorem 8. *Let X be a compact complex* **SKT hyperbolic** *manifold with $dim_{\mathbb{C}} X = n$. Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^{\star}\omega$ the lift to $\widetilde{X}$ of a SKT hyperbolic metric ω on X. Fix a primitive $L^2_{\widetilde{\omega}}$-form ϕ on $\widetilde{X}$ of bidegree (p,q) with $p+q=n-1$ such that*

$$\partial\phi = 0, \qquad \bar{\partial}\phi = 0.$$

Then $\phi = 0$.

Corollary 9. *Let ϕ be a $(n-1,0)$-form (respectively a $(0,n-1)$-form) on a connected complete manifold $(\widetilde{X},\widetilde{\omega})$ such that*

$$\phi \in L^2(\widetilde{X}), \qquad \partial\phi = 0, \qquad \bar{\partial}\phi = 0.$$

If $\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta$ where α and β are bounded 1-forms on $\widetilde{X}$, then

$$\phi = 0.$$

Theorem 10. *Let (X,ω) be a complete Kähler manifold of dimension $2n$ and $\omega = \partial\alpha + \bar{\partial}\beta$ where α and β are respectively a bounded $(0,1)$ and $(1,0)$ forms on X. Then every L_2-form Ψ on X of degree $p \neq m$ satisfies the inequality*

$$\langle \psi, \Delta\psi \rangle \geq \lambda_0^2 \langle \psi, \psi \rangle,$$

where λ_0 is a strictly positive constant which depends only on $n = dimX$, α and β.

Corollary 11. *Let $(\widetilde{X},\widetilde{\omega})$ be a connected complete Kähler manifold. If $\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta$ where α and β are bounded 1-forms on $\widetilde{X}$, then $\mathcal{H}^p_{\Delta_{\widetilde{\omega}}}(\widetilde{X},\mathbb{C}) = 0$, unless $p = n$.*

This is a new conjecture.

Conjecture 12. *If a compact complex manifold admits a balanced hyperbolic metric and an SKT hyperbolic metric, then it admit a Kähler hyperbolic metric.*

References

[1] M. Gromov *Kähler Hyperbolicity and L^2 Hodge Theory J. Diff. Geom.* **33** (1991), 263-292.
[2] S Marouani, D Popovici. *Balanced Hyperbolic and Divisorially Hyperbolic Compact Complex Manifolds* arXiv e-print CV 2107.08972v2, to appear in *Mathematical Research Letters.*
[3] S Marouani. *SKT Hyperbolic and Gauduchon Hyperbolic Compact Complex Manifolds.* arXiv preprint arXiv:2305.08122.