Definition 1. Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and $\omega$ be a metric on $X$: be a $C^\infty$ positive definite $(1, 1)$-form on $X$.

i) $\omega$ is Kähler, if $d\omega = 0$.

ii) $\omega$ is balanced, if $d\omega^{n-1} = 0$.

iii) $\omega$ is Gauduchon, if $\bar{\partial}\partial\omega^{n-1} = 0$, such a metric always exists on a compact complex manifold.

iv) $\omega$ is SKT (or pluriclosed), if $\partial\bar{\partial}\omega = 0$.

Let $\pi_X : \tilde{X} \longrightarrow X$ be the universal cover of $X$ and $\tilde{\omega} = \pi_X^* \omega$ be the Hermitian metric on $\tilde{X}$ that is the lift of $\omega$. Recall that a $C^\infty$ $k$-form $\alpha$ on $X$ is said to be $\bar{\partial}(\text{bounded})$ with respect to $\omega$ if $\pi_X^* \alpha = d\beta$ on $\tilde{X}$ for some $C^\infty (k - 1)$-form $\beta$ on $\tilde{X}$ that is bounded w.r.t. $\tilde{\omega}$. (See [1] and [2]). In general, we propose the following definition which generalizes that of $\bar{\partial}$-bounded of a differential form.

Definition 2. A $C^\infty$ $k$-form $\phi$ on $X$ is said to be $(\bar{\partial} + \partial)$-bounded with respect to $\omega$ if $\pi_X^* \phi = \partial\alpha + \bar{\partial}\beta$ on $\tilde{X}$ for some $C^\infty (k - 1)$-forms $\alpha$ and $\beta$ on $\tilde{X}$ that are bounded w.r.t. $\tilde{\omega}$.

M. Gromov introduced in one of his seminal papers [1] the notion of Kähler hyperbolicity for a compact Kähler manifold $X$. The manifold $X$ is called Kähler hyperbolic if $X$ admits a Kähler metric $\omega$ whose lift $\tilde{\omega}$ to the universal cover $\tilde{X}$ of $X$ can be expressed as

$$\tilde{\omega} = d\alpha$$

for a bounded 1-form $\alpha$ on $\tilde{X}$. As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.

The Kähler hyperbolicity is generalized in [2] to what we call balanced hyperbolicity. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a balanced metric. Meanwhile, a compact complex $n$-dimensional manifold $X$ is said to be balanced hyperbolic if it carries a balanced metric $\omega$ such that $\omega^{n-1}$ is $\bar{\partial}$-bounded. The Brody hyperbolicity is replaced by what we call divisorial hyperbolicity. A compact complex manifold $X$ is called divisorially hyperbolic if there exists no non-trivial holomorphic map from $\mathbb{C}^{n-1}$ to $X$ satisfying certain subexponential volume growth condition.

We introduce the following

Definition 3. Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric $\omega$ on $X$ is said to be

1. **SKT hyperbolic** if $\omega$ is SKT and $(\bar{\partial} + \partial)$-bounded with respect to $\omega$. The manifold $X$ is said to be SKT hyperbolic if it carries a SKT hyperbolic metric.

2. **Gauduchon hyperbolic** if $\omega^{n-1}$ is $(\bar{\partial} + \partial)$-bounded with respect to $\omega$. The manifold $X$ is said to be Gauduchon hyperbolic if it carries a Gauduchon hyperbolic metric.
Lemma 4. The following implication holds:

\( X \text{ is Kähler hyperbolic} \implies X \text{ is SKT hyperbolic} \)

\[ \Downarrow \]

\( X \text{ is balanced hyperbolic} \implies X \text{ is Gauduchon hyperbolic} \)

The following results are taken from [3]

Theorem 5. Every SKT hyperbolic compact complex manifold is Kobayashi hyperbolic.

Remark 6. An immediate observation is that, since a SKT hyperbolic manifold \( X \) contains no rational curves, then by Mori’s cone theorem we get \( K_X \) is nef.

Theorem 7. Every Gauduchon hyperbolic compact complex manifold is divisorially hyperbolic.

Theorem 8. Let \( X \) be a compact complex SKT hyperbolic manifold with \( \dim \mathbb{C} X = n \). Let \( \pi : \widetilde{X} \to X \) be the universal cover of \( X \) and \( \tilde{\omega} := \pi^* \omega \) the lift to \( \widetilde{X} \) of a SKT hyperbolic metric \( \omega \) on \( X \). Fix a primitive \( L^2_\omega \)-form \( \phi \) on \( \widetilde{X} \) of bidegree \((p,q)\) with \( p + q = n - 1 \) such that

\[ \partial \phi = 0, \quad \bar{\partial} \phi = 0. \]

Then \( \phi = 0 \).

Corollary 9. Let \( \phi \) be a \((n - 1,0)\)-form (respectively a \((0,n - 1)\)-form) on a connected complete manifold \((\widetilde{X}, \tilde{\omega})\) such that

\[ \phi \in L^2(\widetilde{X}), \quad \partial \phi = 0, \quad \bar{\partial} \phi = 0. \]

If \( \tilde{\omega} = \partial \alpha + \bar{\partial} \beta \) where \( \alpha \) and \( \beta \) are bounded 1-forms on \( \widetilde{X} \), then

\[ \phi = 0. \]

Theorem 10. Let \((X, \omega)\) be a complete Kähler manifold of dimension 2n and \( \omega = \partial \alpha + \bar{\partial} \beta \) where \( \alpha \) and \( \beta \) are respectively a bounded \((0,1)\) and \((1,0)\) forms on \( X \). Then every \( L^2 \)-form \( \Psi \) on \( X \) of degree \( p \neq m \) satisfies the inequality

\[ \langle \psi, \Delta \psi \rangle \geq \lambda_0^2 \langle \psi, \psi \rangle, \]

where \( \lambda_0 \) is a strictly positive constant which depends only on \( n = \dim X, \alpha \) and \( \beta \).

Corollary 11. Let \((\widetilde{X}, \tilde{\omega})\) be a connected complete Kähler manifold. If \( \tilde{\omega} = \partial \alpha + \bar{\partial} \beta \) where \( \alpha \) and \( \beta \) are bounded 1-forms on \( \widetilde{X} \), then \( H^p_{\Delta_{\tilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0 \), unless \( p = n \).

This is a new conjecture.

Conjecture 12. If a compact complex manifold admits a balanced hyperbolic metric and an SKT hyperbolic metric, then it admit a Kähler hyperbolic metric.

References