Hölder Continuity of Generalized Harmonic Functions in the Unit Disc

Mohamed Mhamdi
(1215 Tehlepet Kasserine Tunisia)
E-mail: mohamedmhamdi@essths.u-sousse.tn

The main purpose of this talk is to discuss about the membership in Hölder classes for (p,q)-harmonic functions $u = K_{p,q}[f]$ such that their boundaries functions $f \in \Lambda_\beta(\mathbb{T})$.

Consider the second order partial differential operators, studied in [2], of the form

$$L_{p,q} := (1-|z|^2)\partial\overline{\partial} + pz\partial + q\overline{z}\overline{\partial} - pq, \qquad z \in \mathbb{D}, \tag{1}$$

where p,q are real parameters. We say that a function u is (p,q)-harmonic if u is twice continuously differentiable in $\mathbb{D}$ and $L_{p,q}u = 0$.

Let consider the associated Dirichlet boundary value problem of functions u, satisfying the equation $L_{p,q}u = 0$,

$$\begin{aligned} L_{p,q}u &= 0 \quad \text{in } \mathbb{D}, \\ u &= f \quad \text{on } \mathbb{T}. \end{aligned} \tag{2}$$

For $p,q \in \mathbb{R}\backslash\mathbb{Z}^-$ such that $p+q > -1$, the $(p,q)-$harmonic Poisson kernel is defined by

$$K_{p,q}(z) = c_{p,q}\,\frac{(1-|z|^2)^{p+q+1}}{(1-z)^{p+1}(1-\overline{z})^{q+1}}, \quad c_{p,q} = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)},$$

where Γ is the Gamma function.

The Solution of (2) has the following (p,q)-harmonic integral representation of $f \in L^1(\mathbb{T})$ which defined by

$$u(z) = K_{p,q}[f](z) := \frac{1}{2\pi}\int_0^{2\pi} K_{p,q}(ze^{-i\theta})f(e^{i\theta})d\theta, \quad z \in \mathbb{D}. \tag{3}$$

Remark that if $p = q = 0$, then the solution u is the classical harmonic function.

Let us recall the notion of ”Hölder” continuity

Definition 1. For a bounded subset E of the complex plane, let ω be a *majorant*, i.e., a continuous increasing function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(t)/t$ is non-increasing for $t > 0$. For a real or complex valued function f on E we write $f \in \Lambda_\omega(E)$ if there is a constant $C > 0$ such that

$$|f(z_1) - f(z_2)| \le C\omega(|z_1 - z_2|), \qquad z_1, z_2 \in E.$$

If $\omega(t) = t^\beta$, the class is simply denoted by $\Lambda_\beta(E)$ which is commonly referred to as the Hölder class for the set E of order $\beta \in (0,1]$.

The following growth estimate is useful.

Lemma 2. [3] *Let $u \in \mathcal{C}^1(\mathbb{D})$ and ω be a majorant satisfying the Dini condition, that is,*

$$\tilde{\omega}(x) := \int_0^x \frac{\omega(t)}{t}dt < \infty, \quad x > 0.$$

If f satisfies $|\partial u(z)| + |\overline{\partial}u(z)| \le C\frac{\omega(1-|z|^2)}{1-|z|^2}$ for all $z \in \mathbb{D}$, then $u \in \Lambda_{\tilde{\omega}}(\mathbb{D})$.

As a consequence of the previous result we get the first main result

Theorem 3. [1] *Let* $p+q>1$ *and* $0<\beta\leq 1$. *Let* $f\in\Lambda_\beta(\mathbb{T})$ *and set* $u=K_{p,q}[f]$. *It yields*

(1) *If* $p+q\neq\beta-1$, *then* $u\in\Lambda_{\min\{\beta,p+q+1\}}(\mathbb{D})$.

(2) *If* $p+q=\beta-1$, *then* $u\in\Lambda_{\omega_\beta}(\mathbb{D})$, *where* $\omega_\beta(t):=t^\beta\Big(1-\log(t)\Big)$.

In particular $u\in\bigcap\limits_{0<\alpha<\beta}\Lambda_\alpha(\mathbb{D})$.

In particular, for $\beta=1$, and $-1<p+q<0$, we have $u\in\Lambda_{p+q+1}(\mathbb{D})$ and we provide an example where $u\notin\Lambda_1(\mathbb{D})$. This example shows the failure of the stability of Lipschitz continuity in the case $p+q<0$, i.e we provide new examples of functions $f\in\Lambda_1(\mathbb{T})$ such that $u=K_{p,q}[f]\notin\Lambda_1(\mathbb{D})$, when $p+q\in(-1,0]$. For more details we refer the reader to [1].

Example 4. [1] Let consider two cases:

- The case $-1<p+q<0$: let $k\in\mathbb{Z}$,

$$u_k(z):=\frac{1}{2\pi}\int_0^{2\pi}K_{p,q}(ze^{-i\theta})e^{ik\theta}d\theta,\quad z\in\mathbb{D}.$$

- The case $p+q=0$ and $p\neq 0$: let

$$u_0(z):=\frac{1}{2\pi}\int_0^{2\pi}K_{p,q}(ze^{-i\theta})d\theta=c_{p,q}F(-p,-q;1;|z|^2),\quad z\in\mathbb{D}.$$

where F is the Gaussian hypergeometric function. Through these two case we can prove that one of the partial derivative of u_k (resp. u_0) is not bounded, which leads to $u_k\notin\Lambda_1(\mathbb{D})$ (resp. $u_0\notin\Lambda_1(\mathbb{D})$).

Definition 5. A sense-preserving diffeomorphism u is said to be $K-$quasiconformal, if

$$\frac{|\partial u(z)|+|\overline{\partial}u(z)|}{|\partial u(z)|-|\overline{\partial}u(z)|}\leq K,$$

throughout the given region Ω, where $K\in[1,\infty)$ is a constant.

Under an extra condition, we preserve the same order of f and u.

Theorem 6. [1] *Let* $p+q>1$ *and* $0<\beta\leq 1$, $f\in\Lambda_\beta(\mathbb{T})$ *and set* $u=K_{p,q}[f]$, *be a K-quasiconformal mapping. Then* $u\in\Lambda_\beta(\mathbb{D})$.

References

[1] Khalfallah A, Mhamdi, M. *Hölder Continuity of Generalized Harmonic Functions in the Unit Disc. Complex Analysis and Operator Theory*, 16(7), 101.(2022)

[2] Klintborg M, Olofsson A, *A series expansion for generalized harmonic functions*, *Anal. Math. Phys.* **11** no. 3, Paper No. 122 (2021)

[3] Pavlovič M, *Introduction to function spaces on the disk*. Posebna Izdanja 20. Matematicki Institut SANU, Belgrade, 2004.