How far apart can the projection of the centroid of a convex body and the centroid of its projection be?

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Let K be a convex body in $\mathbb{R}^n$, i.e., a compact convex set with non-empty interior. The centroid (the center of mass) of K is the point

$$c(K) = \frac{1}{|K|}\int_K x\,dx,$$

where $|K|$ denotes the volume of K and the integration is with respect to Lebesgue measure.

In this work we study the following question. Let H be a hyperplane in $\mathbb{R}^n$. Denote by $P_H c(K)$ the orthogonal projection of the centroid of K onto H and by $c(P_H K)$ the centroid of the projection of K onto H. For centrally symmetric bodies these two points coincide, but for non-symmetric bodies these points are generally different. Thus it is natural to ask how far apart these two points can be relative to some linear size of K. More precisely, we are interested in the smallest constant D_n such that for any convex body K in $\mathbb{R}^n$ we have

$$|P_H c(K) - c(P_H K)| \le D_n w_K(u),$$

where u is the unit vector parallel to the segment connecting $P_H c(K)$ and $c(P_H K)$, and $w_K(u)$ is the width of K in the direction of u, given by

$$w_K(u) = \max_{x\in K}\{\langle x,u\rangle\} - \min_{x\in K}\{\langle x,u\rangle\}.$$

Questions of this type began attracting attention several years ago in connection to Grünbaum-type inequalities for sections and projections; see [5], [2]. In particular, an analogue of the question above for sections of convex bodies is stated in [4, p. 127]. For other questions related to distances between various centroids the reader is referred to the book [1, p. 36] and the references contained therein.

Theorem 1 ([3]). *Let D_n, $n \ge 3$, be the smallest number such that*

$$|P_H c(K) - c(P_H K)| \le D_n \cdot w_K(u), \tag{1}$$

for every convex body K in $\mathbb{R}^n$ and every hyperplane $H \subset \mathbb{R}^n$, where

$$u = \frac{P_H c(K) - c(P_H K)}{|P_H c(K) - c(P_H K)|},$$

provided $P_H c(K) \ne c(P_H K)$. Then

(i) $D_3 = 1 - \sqrt{\frac{2}{3}} \approx 0.1835$*; the sequence* $\{D_n\}_{n=3}^{\infty}$ *is increasing; and* $\lim_{n\to\infty} D_n \approx 0.2016$.

(ii) *Inequality* (1) *turns into equality if and only if K is a body obtained as follows. For a fixed hyperplane H and a vector u parallel to H, denote by θ a unit normal vector to H and take any $(n-2)$-dimensional subspace U orthogonal to u and transversal to θ. Let L_0 be any convex body in U. Denote by tL_0 the dilation of L_0 with respect to its centroid by a factor of $t = t_{max}$, which will be defined later in the proof. Let λ, μ, ν be real numbers, $\mu \neq \nu$. Define $L_1 = tL_0 + \lambda u + \mu\theta$ and $L_2 = tL_0 + \lambda u + \nu\theta$. Then K is the convex hull of L_0, L_1, and L_2. Figure 1 shows an example of such a body in $\mathbb{R}^3$ when $H = \{x_3 = 0\}$, $u = e_1$, and U is the linear span of e_2.*

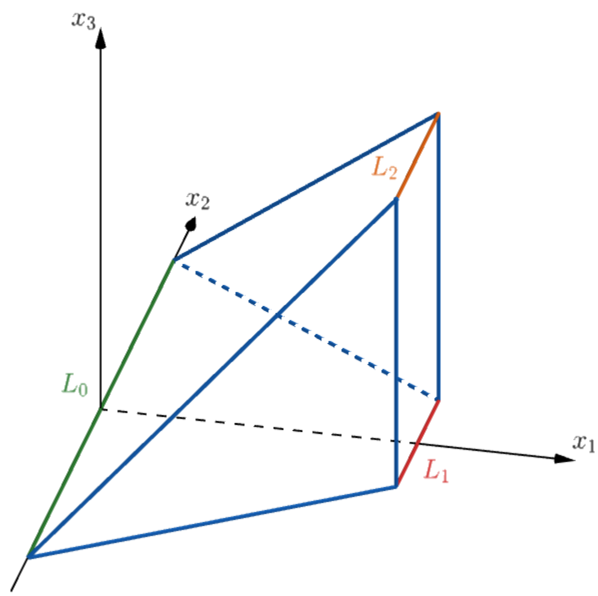

FIGURE 1.

References

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[2] S. Myroshnychenko, M. Stephen, and N. Zhang, *Grünbaum's inequality for sections*, J. Funct. Anal. **275** (2018), no. 9, 2516–2537.

[3] S. Myroshnychenko, K. Tatarko, V. Yaskin, *How far apart can the projection of the centroid of a convex body and the centroid of its projection be?*, pre-print arXiv:2212.14456

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[5] M. Stephen and N. Zhang, *Grünbaum's inequality for projections*, J. Funct. Anal. **272** (2017), no. 6, 2628–2640.