Contractions of representations and realizations of Lie algebras

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Realizations (first-order differential operators) and representations (linear operators) of Lie algebras are widely applicable in modern group analysis of differential equations, in classification of gravity fields, in geometric control theory, in difference schemes for numerical solutions of differential equations, in theory of invariants, etc.

To study limit processes that connect different theories or their mathematical models it is useful to investigate contractions (limit connections) of their underling symmetries. In practice, we first study possible limit processes between abstract Lie algebras, and then, we need to find a way how to introduce similar limits in the existing realizations or representations of Lie algebras. Unfortunately, the direct application of the known contraction to a realization or representation of a Lie algebra gives several zero operators, what makes it impossible for further application to real equations.

To overcome this obstacle, we propose to construct a parameterized series of realizations and representations based on the action of the contraction matrix on the tensor of structure constants. The realizations and representations obtained in this way coincide in the limit with the corresponding realizations and representations of contracted Lie algebras. We provide the algorithm for constructing parameterized series and present a number of illustrative examples.

For clarity, let's consider main definitions. Let $\mathcal{L}_n(V)$ be the variety of n-dimensional Lie algebras (set of Lie brackets) on a vector space V over the field $\mathbb{R}$, then each n-dimensional Lie algebra $\mathfrak{g} = (V, [.,.])$ corresponds to a multiplication rule $\mu \in \mathcal{L}_n$: $\forall\ x, y \in V \quad [x, y] = \mu(x, y)$.

General linear group $\mathrm{GL}(V)$ acts on the variety of Lie brackets as follows:

$$\forall A \in \mathrm{GL}(V),\ \forall \mu \in \mathcal{L}_n \quad (A\mu)(x, y) = A^{-1}(\mu(Ax, Ay)) \quad \forall x, y \in V.$$

Consider a continuous function $U(\varepsilon) = U\colon (0, 1] \to \mathrm{GL}(V)$ and a parameterized family of Lie algebras $\mathfrak{g}_\varepsilon = (V, [.,.]_\varepsilon)$ with the Lie product defined for arbitrary elements of the vector space $[x, y]_\varepsilon = {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$. All such algebras are isomorphic to the initial algebra $\mathfrak{g} = (V, [.,.])$.

Definition 1. If $\forall x, y \in V$ there exists a limit

$$[x, y]_0 := \lim_{\varepsilon\to+0} [x, y]_\varepsilon = \lim_{\varepsilon\to+0} {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$$

then $[.,.]_0$ is a well-defined Lie bracket and Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *contraction* of the Lie algebra $\mathfrak{g}$.

Let $M \subset \mathbb{R}^m$ be an open domain. Let us denote the Lie algebra of smooth vector fields on M by $\mathrm{Vect}(M)$.

Definition 2. A *realization* of a Lie algebra $\mathfrak{g}$ in vector fields on M is a homomorphism

$$R\colon \mathfrak{g} \to \mathrm{Vect}(M).$$

Let us consider the algebra of endomorphisms $gl(V)$ of the vector space V and define a representation, which is closely related to Lie algebra module.

Definition 3. A *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism

$$\varphi\colon \mathfrak{g} \to gl(V).$$

Let us outline the algorithm in the case of realizations:

(1) Construct parameterized structure constants using the continuous function U, that do realize the desired contraction $C^{k'}_{\varepsilon,i'j'} := (U_\varepsilon)^i_{i'}(U_\varepsilon)^j_{j'}(U_\varepsilon{}^{-1})^{k'}_k C^k_{ij}$, where C^k_{ij} are structure constants of the initial Lie algebra.

(2) Calculate ε-dependent adjoint actions (using the structure constants $C^{k'}_{\varepsilon,i'j'}$), exponents and differential 1-forms: $\mathrm{ad}^\varepsilon e_i$, $\exp(-x_i \mathrm{ad}^\varepsilon e_i)$, $\omega^\varepsilon(x)$.

(3) Find the inverse transformation to obtain the vector fields $\xi^\varepsilon(x) = (\omega^\varepsilon(x))^{-1}$, that are the parameterized realization that do contracts to the realization of the contracted Lie algebra.

To conclude let us mention that contraction of the fixed realization or representation of a Lie algebra is more complicated task. Namely, in the case of realization, we first have to define it's subalgebra (studying the kernel of the linear operator in the initial point), then we have to find the equivalence transformations to the canonical realization. After that we can apply our algorithm and complete it by the inverse of the equivalence transformations.

References

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